MATH 249 by Oliver Pechenik

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Contents

1 Enumeration

Definition 1.1: Bijection

For two sets A and B and let $f : A \to B$.

1. f is **surjective** (onto) if for each $b \in B$, there exists $a \in A$ such that $f(a) = b$.
2. f is **injective** (one-to-one) if $f(a) = f(a')$ implies $a = a'$.
3. f is **bijective** if it is injective and surjective.

Proposition 1.1

$f : A \to B$ is a bijection if and only if f as an inverse.

Corollary 1.1

If there exists a bijection $f : A \to B$, then $|A| = |B|$.

Definition 1.2:

We write

$$[n] = \{1, 2, \dots, n\}$$

Definition 1.3: Permutation

A **permutation** of n is a bijection $[n] \to [n]$. We write S_n for the set of permutations of n. S_n is known as a **symmetric group**.

Example 1.1: 2-line Notation and 1-line Notation

The following

$$u(1) = 3 \quad u(2) = 1 \quad u(3) = 2$$

is a permutation in S_3. We could write it as

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

which is known as a **2-line notation**. Alternatively, we could also just write

$$312$$

which is the **1-line notation**.

Example 1.2

We have

$$S_3 = \{123, 132, 213, 231, 312, 321\}$$

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Proposition 1.2

We have

$$|S_n| = n! \qquad \text{where} \qquad k! = \prod_{i=1}^{k} i$$

Proof. □

1.1 Binomial Coefficient

Definition 1.4: Binomial coefficient

For a set S, write $\binom{S}{k}$ for the collection of all k-element subset of S. We call it "S choose k". We define

$$\binom{n}{k} = \left|\binom{[n]}{k}\right|$$

Example 1.3

As an example, we have

$$\binom{[4]}{2} = \Big\{\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}\Big\}$$

Proposition 1.3

For $0 \leq k \leq n$,

$$\binom{n}{k} = \binom{n}{n-k}$$

Proof. (Choosing k things is the same as not choosing $n - k$ things. —Suno)
Want a bijection:

$$f : \binom{[n]}{k} \to \binom{[n]}{n-k}$$

For $S \in \binom{[n]}{k}$, define $f(S) = [n] \backslash S$, then $f(S) \in \binom{[n]}{n-k}$. Easy to see that this is a bijection. □

Proposition 1.4: Fibre

If $f : A \to B$ is a surjection with $|f^{-1}(b)| = k$ for all $b \in B$, then

$$|A| = k \cdot |B|$$

where $f^{-1}(b)$ is called the **fibre over** b .

Proposition 1.5

We have

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Proof. STP

$$n! = \binom{n}{k} k!(n-k)!$$

Consider $f : S_n \to \binom{[n]}{k}$ defined by $f(a_1 a_2 \dots a_n) = \{a_1, a_2, \dots, a_k\}$. This is clearly a surjection. Each fibre has size of $k!(n-k)!$. Therefore, we conclude

$$n! = \binom{n}{k} k!(n-k)!$$

□

1.2 Pascal Recurrence

Theorem 1.1: Pascal's Recurrence

We have

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Proof. Want a bijection

$$f : \binom{[n]}{k} \to \binom{[n-1]}{k-1} \sqcup \binom{[n-1]}{k}$$

Define $f(S)$ as following:

$$f(S) = \begin{cases} S & \text{if } n \notin S \\ S \backslash \{n\} & \text{otherwise} \end{cases}$$

Clearly the function lands in the right place (so it is injective). It is easy to check that this is also a surjection, thus bijection. □

Proposition 1.6

We have
$$\sum_{i=0}^{n} \binom{n}{i} = 2^n$$

Proof. Write 2^S for the collection of all subsets of S. Observe that
$$2^{[n]} = \binom{[n]}{0} \sqcup \binom{[n]}{1} \sqcup \cdots \sqcup \binom{[n]}{n}$$

Hence it is enough to enumerate $2^{[n]}$. Let $\{0,1\}^n$ be the set of all n-tuples $(v_1, \ldots, v_n)$ with each $v_i \in \{0,1\}$ (binary string of length n). Note
$$|\{0,1\}^n| = 2^n$$

Finally, we have a bijection
$$f : 2^{[n]} \to \{0,1\}^n \quad \text{given by } (f(S))_i = \begin{cases} 0 & \text{if } i \notin S \\ 1 & \text{if } i \in S \end{cases}$$

There is an inverse, so this is a bijection.

Remark: S is an arbitrary subset of $[n]$, function f is a function that turns the set S into a binary string.

$\square$

2 Generating Series

The idea: If you have a sequence of numbers,

$$a_0, a_1, a_2, \dots$$

instead look at the power series

$$a_0 + a_1 x + a_2 x^2 + \dots = A(x)$$

This is a remarkably useful change of perspective.

Example 2.1

Suppose we have a sequence: $a_i = 1$ for all i, then

$$A(x) = 1 + x + x^2 + x^3 + \cdots$$

Consequently,

$$xA(x) = x + x^2 + x^3 + \cdots$$

Subtract them we get $A(x) - xA(x) = 1 = (1-x)A(x)$, so $A(x) = \frac{1}{1-x}$.

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2.1 Formal Power Series

Definition 2.1: Formal Power Series

A **FPS** is an infinite sequence of numbers. We add FPSs by

$$A + B = a_0 + b_0, a_1 + b_1, a_2 + b_2, \dots$$

and multiply by

$$AB = a_0 b_0, a_0 b_1 + a_1 b_0, a_0 b_2 + a_1 b_1 + a_2 b_0, \dots$$

FPS form a **ring**, which means you can add, subtract, multiply just like in $\mathbb{Z}$.

Definition 2.2: Polynomial

A **polynomial** is a FPS such that for some $N \in \mathbb{N}$ we have $a_i = 0 \quad \forall\, i \geq N$.

2.1.1 Binomial Theorem

Theorem 2.1: Binomial Theorem

We have
$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

Proof. Recall the bijection $f : 2^{[n]} \to \{0,1\}^n$ given by
$$(f(S))_i = \begin{cases} 0 & \text{if } i \notin S \\ 1 & \text{if } i \in S \end{cases}$$

$$S \longleftrightarrow (v_1, \ldots, v_n) \in \{0,1\}^n$$
$$|S| \longleftrightarrow v_1 + \cdots + v_n = \sum f(S)$$

$$\begin{aligned}
\sum_{k=0}^{n} \binom{n}{k} x^k = \sum_{S \in 2^{[n]}} x^{|S|} = \sum_{S \in 2^{[n]}} x^{\sum f(S)} &= \sum_{v \in \{0,1\}^n} x^{v_1 + \cdots + v_n} \\
&= \left(\sum_{v_1 \in \{0,1\}} x^{v_1} \right) \cdots \left(\sum_{v_n \in \{0,1\}} x^{v_n} \right) \\
&= \left(\sum_{v \in \{0,1\}} x^{v} \right)^n \\
&= (1+x)^n
\end{aligned}$$

Remark: Clarification for the first equation: For each term with exponent (power) of $|S|$, there are exactly $\binom{n}{|S|}$ of them, which yields us the coefficient of $\binom{n}{k}$ on the left hand side. Therefore, looping thru all the possible subsets of $2^{[n]}$ is the same as looping k through 0 to n.

$\square$

Definition 2.3: Generating Series

Given a set S and a **statistic/ weight function**
$$\xi : S \to \mathbb{N}$$
the **generating series** is $\sum_{a \in S} x^{\xi(a)}$. It is erquired that we **do not** have infinite objects of the same weight, that is, the fibre above each natural number is finite: $|\xi^{-1}(k)| < \infty \quad \forall\, k \in \mathbb{N}$.

2.1.2 Weight Function: des and Des

Definition 2.4: Descent

Given a permutation v, we break it into max increasing runs. Let $\mathrm{des}(v)$ be defined as

$$\mathrm{des}(v) = \#\text{ of bars } = \#\text{ of runs } - 1$$

A **descent** of $v \in S_n$ is an $i \in [n-1]$ with $v(i) > v(i+1)$. We write $\mathrm{Des}(v)$ for the set of descents. Then

$$|\mathrm{Des}(v)| = \mathrm{des}(v)$$

Definition 2.5: Eulerian Polynomial

The **Eulerian Polynomial** $S_n(t)$ is the generating function for S_n by des.

Example 2.2

$S_1(t) = 1$, $S_2(t) = 1 + x$, $S_3(t) = 1 + 4t + t^2$, and $S_4(t) = 1 + 11t + 11t^2 + t^3$. Use $S_3(t)$ as an example to illustrate the idea: We know that

$$S_3 = \{123, 132, 213, 231, 312, 321\}$$

Notice that we have:

$\mathrm{des} = 0$**:** 123

$\mathrm{des} = 1$**:** 132, 213, 231, 312

$\mathrm{des} = 2$**:** 321

Thus the corresponding coefficients are 1, 4, and 1.

Definition 2.6: Eulerian Numbwer

The **Eulerian number** $\left\langle {n \atop k} \right\rangle$ is the number of permutations of n with k descents.

Result 2.1

There is no elementary formula for $\left\langle {n \atop k} \right\rangle$. We have

$$\left\langle {n \atop k} \right\rangle = (n-k)\left\langle {n-1 \atop k-1} \right\rangle + (k-1)\left\langle {n-1 \atop k} \right\rangle$$

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Definition 2.7: Multiset

Fix $t \geq 1$. Let $M(t)$ be collection of all **multisets** of arbitary size with t types.

Discovery 2.1

Multisets are in bijection with weak composition

$$\{(m_1, m_2, \ldots, m_t) : m_i \in \mathbb{N}\}$$

Let m_i be the number of objects of type i.

Result 2.2

We basically treat multisets and weak compositions interchangeably.

Definition 2.8: Size of Multiset

Define the **size** of a multiset to be

$$|(m_1, \ldots, m_t)| = m_1 + m_2 + \cdots + m_t$$

Theorem 2.2: How many multisets in $M(t)$ of size n?

$M(t)$ contains $\binom{n+t-1}{t-1} =: \left(\!\binom{t}{n}\!\right)$ elements of size n.

Proof. See https://en.wikipedia.org/wiki/Stars_and_bars_(combinatorics).
Draw n dots. Add vertical bars to separate the dots into blocks of $m_1, m_2, \ldots, m_t$ consecutive dots. Such diagrams with $(t-1)$ bars are in bijection with the multisets we want to count. But these look like binary strings, so there are $\binom{n+t-1}{t-1}$ of them. □

2.1.3 Negative Binomial Theorem

Theorem 2.3: Negative Binomial Theorem

We have

$$\sum_{n=0}^{\infty} \binom{n+t-1}{t-1} x^n = \frac{1}{(1-x)^t} = (1-x)^{-t}$$

Proof.

$$\begin{aligned}
\sum_{n=0}^{\infty} \binom{n+t-1}{t-1} x^n &= \sum_{n=0}^{\infty} (\#\text{ of size } n \text{ multisets of } t \text{ types})\, x^n \\
&= \sum_{(m_1,\dots,m_t)\in M(t)} x^{m_1+\cdots+m_t} \\
&= \sum_{(m_1,\dots,m_t)\in \mathbb{N}^t} x^{m_1+\cdots+m_t} \\
&= \left(\sum_{m_1=0}^{\infty} x^{m_1}\right) \cdots \left(\sum_{m_t=0}^{\infty} x^{m_t}\right) \\
&= \left(\sum_{m=0}^{\infty} x^{m}\right)^t \\
&= \left(\frac{1}{1-x}\right)^t
\end{aligned}$$

□

Definition 2.9: Generating Serires Notation

Given a set $\mathcal{A}$ with weight function $\omega : \mathcal{A} \to \mathbb{N}$, let's write

$$\Phi_{\mathcal{A}}(x) = \sum_{\alpha\in\mathcal{A}} x^{\omega(\alpha)}$$

for the generating series. We also write

$$\mathcal{A}_n = \omega^{-1}(n) = \{\alpha \in \mathcal{A} : \omega(\alpha) = n\}$$

2.1.4 Sum Lemma

Lemma 2.1: Sum Lemma

Let $\mathcal{A}, \mathcal{B}$ be *disjoint* sets and let $\omega : \mathcal{A} \sqcup \mathcal{B} \to \mathbb{N}$ be a weight function. Then ω is a weight function on $\mathcal{A}$ and $\mathcal{B}$ separately. Moreover,

$$\Phi_{\mathcal{A}\sqcup\mathcal{B}} = \Phi_{\mathcal{A}} + \Phi_{\mathcal{B}}$$

Proof. From the definition of generating series,

$$\Phi_{\mathcal{A}\sqcup\mathcal{B}} = \sum_{\alpha\in\mathcal{A}\sqcup\mathcal{B}} x^{\omega(\alpha)} = \sum_{\alpha\in\mathcal{A}} x^{\omega(\alpha)} + \sum_{\alpha\in\mathcal{B}} x^{\omega(\alpha)} = \Phi_{\mathcal{A}}(x) + \Phi_{\mathcal{B}}(x)$$

(The condition that $\mathcal{A} \cap \mathcal{B} = \varnothing$ is needed for the second equality.) □

2.1.5 Product Lemma

Lemma 2.2: Product Lemma

Let $\mathcal{A}, \mathcal{B}$ have weight functions $\omega : \mathcal{A} \to \mathbb{N}$ and $\nu : \mathcal{B} \to \mathbb{N}$. Define $\eta : \mathcal{A} \times \mathcal{B} \to \mathbb{N}$ by

$$\eta(\alpha, \beta) = \omega(\alpha) + \nu(\beta)$$

Then η is weight function, and

$$\Phi^{\eta}_{\mathcal{A}\times\mathcal{B}}(x) = \Phi^{\omega}_{\mathcal{A}}(x) \cdot \Phi^{\nu}_{\mathcal{B}}(x)$$

(The superscripts ω, ν, and η indicate which weight function is being used for each set.)

Proof. To see that η is a weight function, consider any $n \in \mathbb{N}$. There are $n+1$ choices for an integer $0 \leq k \leq n$. For each such k, there are only finitely many pairs $(\alpha, \beta) \in \mathcal{A} \times \mathcal{B}$ with $\omega(\alpha) = k$ and $\omega(\beta) = n - k$. That is, the set of elements of $\mathcal{A} \times \mathcal{B}$ of weight n is

$$(\mathcal{A} \times \mathcal{B})_n = \bigcup_{k=0}^{n} \mathcal{A}_k \cdot \mathcal{B}_{n-k}$$

a finite (disjoint) union of finite sets. It follows that there are only finitely many elements of $\mathcal{A} \times \mathcal{B}$ of weight n. Now,

$$\begin{aligned}\Phi^{\eta}_{\mathcal{A}\times\mathcal{B}}(x) &= \sum_{(\alpha,\beta)\in\mathcal{A}\times\mathcal{B}} x^{\eta(\alpha,\beta)} = \sum_{\alpha\in\mathcal{A}} \sum_{\beta\in\mathcal{B}} x^{\omega(\alpha)+\nu(\beta)} \\ &= \sum_{\alpha\in\mathcal{A}} x^{\omega(\alpha)} \cdot \sum_{\beta\in\mathcal{B}} x^{\nu(\beta)} = \Phi^{\omega}_{\mathcal{A}}(x) \cdot \Phi^{\nu}_{\mathcal{B}}(x)\end{aligned}$$

□

Let $\mathcal{A}$ have a weight function ω. Define a weight function on $\mathcal{A}^k$ by

$$\omega_k(\alpha_1, \ldots, \alpha_k) = \sum_i \omega(\alpha_i)$$

Corollary 2.1

By Product Lemma,

$$\Phi_{\mathcal{A}^k}(x) = (\Phi_{\mathcal{A}}(x))^k$$

Example 2.3

The generating function for $\{0, 1\}^n$ is $(1 + x)^n$ and that for $\mathbb{N}^n$ is $\left(\frac{1}{1-x}\right)^n$.

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Definition 2.10:

Let

$$\mathcal{A}^* = \bigcup_{k=0}^{\infty} \mathcal{A}^k$$

Define $\omega^* : \mathcal{A}^* \to \mathbb{N}$ by $\omega^*(\alpha_1, \ldots, \alpha_k) = \omega_k(\alpha_1, \ldots, \alpha_k) = \sum_{i=1}^{k} \omega(\alpha_i)$.

Lemma 2.3

ω^* is a weight function on $\mathcal{A}^*$ if and only if $\mathcal{A}_0 = \varnothing$.

Proof. Suppose $\alpha \in \mathcal{A}$ has $\omega(\alpha) = 0$, then every $(\alpha, \alpha, \ldots, \alpha)$ has weight 0. Since we cannot have infinitely many elements with the same weight, so this is impossible. Conversely, suppose $\omega(\alpha) > 0$ for all $\alpha \in \mathcal{A}$. Then each element of $\mathcal{A}^k$ has weight $\geq k$. Hence the elements of $\mathcal{A}^*$ has weight $n \leq$ the elements of $\mathcal{A}^*$ of length at most n. Therefore,

$$(\mathcal{A}^*)_n \subseteq \left(\bigcup_{k=0}^{n} \mathcal{A}^k \right)_n = \bigcup_{k=0}^{n} \left((\mathcal{A}^k)_n \right)$$

which is finite, so ω^* is a weight function. $\square$

2.2 String Lemma

Theorem 2.4: String Lemma

Let $\mathcal{A}$ have weight function ω with $\mathcal{A}_0 = \varnothing$. Then

$$\Phi_{\mathcal{A}^*}(x) = \frac{1}{1 - \Phi_{\mathcal{A}}(x)}$$

Proof. We have

$$\Phi_{\mathcal{A}^*}(x) = \sum_{k=0}^{\infty} \Phi_{\mathcal{A}^k}(x) = \sum_{k=0}^{\infty} (\Phi_{\mathcal{A}}(x))^k = \frac{1}{1 - \Phi_{\mathcal{A}}(x)}$$

where the first equation is sum law, second equation is product law, and the third equation is geometric series. $\square$

Discovery 2.2

How many binary strings of length n?

Proof. We define $\omega(0) = 1$ and $\omega(1) = 1$, then

$$\Phi_{\{0,1\}}(x) = 2x$$

then $\{0,1\}^*$ is all binary strings, so

$$\Phi_{\{0,1\}^*}(x) = \frac{1}{1-\Phi_{\{0,1\}}(x)} = \frac{1}{1-2x} = 1 + 2x + 4x^2 + 8x^3 + \cdots$$

□

Example 2.4

Let's make a new weight function ν^* with $\nu^*(\alpha_1, \alpha_2, \ldots, \alpha_k) = \#0 + 5 \cdot \#1$. This comes from the weight function

$$\begin{aligned} \nu : \{0,1\} &\to \mathbb{N} \\ 0 &\mapsto 1 \\ 1 &\mapsto 5 \end{aligned}$$

How many binary strings of ν^*-cost n?

Proof. We have

$$\Phi_{\{0,1\},\nu}(x) = x + x^5 \qquad \text{and} \qquad \Phi_{\{0,1\}^*,\nu^*}(x) = \frac{1}{1-(x+x^5)}$$

the answer would be the coefficient of the term with exponent n. □

2.3 Composition

Definition 2.11: Composition

A **composition** is a finite sequence of positive integers

$$\gamma = (c_1, \ldots, c_k)$$

with $k \in \mathbb{N}$ and each $c_i > 0$. The **length** is $k = \ell(\gamma)$, which is the number of parts. The **size** of γ is $|\gamma| = c_1 + \cdots + c_k$.

Example 2.5

The compositions of 3 are: 3, 12, 21, 111.

Observe that the set of all components is $\mathcal{C} = \mathbb{P}^*$. (the set of positive integers.)

Theorem 2.5

The generating series for $\mathcal{C}$ with respect to size is

$$\Phi_{\mathcal{C}}(x) = 1 + \frac{x}{1-2x}$$

Indeed, the number of compositions of size n is

$$\mathcal{C} = \begin{cases} 1 & \text{if } n = 0 \\ 2^{n-1} & \text{if } n \geq 1 \end{cases}$$

Proof. We have

$$\Phi_{\mathbb{P}}(x) = \sum_{n=1}^{\infty} x^n = x \sum_{n=0}^{\infty} x^n = x \cdot \frac{1}{1-x}$$

By String Lemma,

$$\Phi_{\mathcal{C}}(x) = \Phi_{\mathbb{P}^*}(x) = \frac{1}{1 - \frac{x}{1-x}} = 1 + \frac{x}{1-2x}$$

By geometric series,

$$1 + \frac{x}{1-2x} = 1 + \sum_{k=0}^{\infty} 2^k x^{k-1} = 1 + \sum_{n=1}^{\infty} 2^{n-1} x^n$$

Now $[x^n]\Phi_{\mathcal{C}}(x)$ is in the form we wanted. $\square$

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Theorem 2.6

$|\mathcal{C}_n| = 2^{n-1}$ for $n > 0$.

Proof. This is a bijective proof.
Define $f : \mathcal{C} \to 2^{[\mathbb{P}]}$ by

$$\gamma = (c_1, c_2, \dots, c_k) \mapsto_f \{c_1, c_1 + c_2, c_1 + c_2 + c_3, \dots, c_1 + c_2 + \cdots + c_{k-1}\}$$

Notice that the function only functions when $k > 0$ (otherwise we would need c_{-1}, which is inapplicable). However, if $\gamma \in \mathcal{C}_n$ for $n > 0$, then $f(\gamma)$ makes sense. Note that each element $f(\gamma)$ is at least c_1, which has a value of at least 1. Moreover, every element of $f(\gamma)$ is at most $c_1 + \cdots + c_{k-1}$, which has a value of at most $n - 1$ (recall that this sum is also equal to $n - c_k$). Therefore, $f(\gamma) \in 2^{[n-1]}$. We now claim that the function f is a bijection.

$$f : \mathcal{C}_n \to 2^{[n-1]}$$

This inverse map is

$$g : 2^{[n-1]} \to \mathcal{C}_n$$

given as: for $S \in 2^{[n-1]}$, write the elemetns in increasing order,

$$1 \leq S_1 < S_2 < \cdots < S_j \leq n - 1$$

Define $g(S) = (S_1, S_2 - S_1, \ldots, S_j - S_{j-1}, n - S_j)$. Then you can check that $f(g(S)) = S$ and $g(f(\gamma)) = \gamma$. $\square$

Corollary 2.2

We have

$$\texttt{number of compositions of size } n \texttt{ and length } k =$$
$$\texttt{number of subsets of } [n-1] \texttt{ of size } k-1 = \binom{n-1}{k-1}$$

2.3.1 Examples for Generating Functions

Example 2.6

Compositions with each part 1 or 2. That is, we consider $\{1,2\}^*$. Writing down the generating function we have

$$\Phi_{\{1,2\}^*}(x) = \frac{1}{1 - \Phi_{\{1,2\}}(x)} = \frac{1}{1 - (x + x^2)}$$

Easy observation we find

size		number
2	$\{11\}, \{2\}$	2
3	$\{12\}, \{21\}, \{111\}$	3
4	$\{22\}, \{211\}, \{121\}, \{112\}, \{1111\}$	5
5	$\ldots$	8

Notice that the numbers form a Fibonacci sequence.

Example 2.7

Compositions with each part of size at least 2. Consider $\{2,3\ldots\}^*$. Thus we have

$$\Phi_{\{2,3,\ldots\}^*}(x) = \frac{1}{1 - \Phi_{\{2,3,\ldots\}}(x)} = \frac{1}{1 - \left(\frac{x^2}{1-x}\right)} = 1 + \frac{x^2}{1 - x - x^2}$$

Example 2.8

Compositions with odd parts only. Consider $\{1,3,5,\ldots\}^*$.

$$\Phi_{\{1,3,5,\ldots\}}(x) = x + x^3 + x^5 + \cdots = x\left(1 + x^2 + x^4 + \cdots\right) = x \cdot \frac{1}{1 - x^2}$$

Therefore, by String Lemma,

$$\Phi_{\{1,3,5,\ldots\}^*}(x) = \frac{1}{1 - x \cdot \frac{1}{1-x^2}} = 1 + \frac{x}{1 - x - x^2}$$

2.4 Permutations

2.4.1 Weight Function: maj

Definition 2.12: Major Index

For $\omega \in S_n$, $\sum \mathrm{Des}(\omega) = \mathrm{maj}(\omega)$ is the **major index**.
(In honour of Major Percy MacMahon).

Example 2.9

Suppose we have a permutation $u = 1384625$, so we have $\mathrm{Des}(u) = \{3, 6\}$. Consequently, we have $\mathrm{maj}(u) = 3 + 6 = 9$.

Result 2.3

For $\omega \in S_n$, $\mathrm{des}(\omega) \leq \mathrm{maj}(\omega)$.

Theorem 2.7:

We have

$$\sum_{\omega \in S_n} q^{\mathrm{maj}(\omega)} = [n]_q!$$

which is called a **quantum factorial**, or **q-factorial**.

Lecture 7 - Wednesday, September 18

Given $u \in S_{n-1}$, we can produce $\omega \in S_n$ by adding n to the 1-line notation for u in any of the n positions. We wonder what happens to maj.

Example 2.10

Consider $u = 132 \in S_3$. Then $\mathrm{maj}(u) = 2$. Then we would have

$$\begin{array}{r|l} 4132 & \mathrm{maj} = 4 \\ 1432 & \mathrm{maj} = 5 \\ 1342 & \mathrm{maj} = 3 \\ 1324 & \mathrm{maj} = 2 \end{array}$$

Discovery 2.3

Our claim is that those n permutations have possible values of

$$\mathrm{maj}(u), \mathrm{maj}(u) + 1, \ldots, \mathrm{maj}(u) + n - 1$$

each of which shows up exactly once.

Proof. This is a proof by example:
Consider $u = 13846725 \in S_8$, then we have

13846725 9	maj = 9
138467 9 25	maj = 10
138 9 46725	maj = 11
9 13846725	maj = 12

So far the number 9 has been moving to the right the whole time, it is time for it to walk back to the right.

1 9 3846725	maj = 13
13 9 846725	maj = 14
1384 9 6725	maj = 15
13846 9 725	maj = 16
1384672 9 5	maj = 17

Notice that the "algorithm" is that we first put n in all the descent positions (including last position) from right to left. Then we visit all the other positions from left to right. ☐

Result 2.4

Hence from the discovery abvoe, we find

$$\sum_{\omega \in S_n} q^{\mathrm{maj}(\omega)} = \sum_{u \in S_{n-1}} q^{\mathrm{maj}(u)} \left(1 + q + \cdots + q^{n-1}\right) = \left(1 + q + \cdots + q^{n-1}\right) \cdot \sum_{u \in S_{n-1}} q^{\mathrm{maj}(u)}$$

where we call $\left(1 + q + \cdots + q^{n-1}\right) = [n]_q$, which is called the **quantum integer**, or the **q-integer**.

Definition 2.13: Mahonian

A statistic ξ on permutation is Mahonian if

$$\sum_{\omega \in S_n} q^{\xi(\omega)} = [n]_q!$$

Corollary 2.3

maj is Mahonian.

2.4.2 Weight Function: inv

Definition 2.14: Inversion

An **inversion** of $\omega \in S_n$ is a pair (i, j) with $1 \le i < j \le n$ but $\omega(i) > \omega(j)$. Consequently, the **inversion number** of ω is the number of inversion.

Example 2.11

Consider 13846725, it has the inversion number of value 11.

Theorem 2.8

inv is Mahonian:

$$\sum_{\omega \in S_n} q^{\mathrm{inv}(\omega)} = [n]_q!$$

Lecture 8 - Friday, September 20

Proof. We build $\omega \in S_n$ from $u \in S_{n-1}$ by adding n into the 1-line notation for u in any of the n spots. We claim that the number of possible inv's for the new permutation are $\mathrm{inv}(u), \ldots, \mathrm{inv}(u) + n - 1$. This is indeed true because the number of extra inversions is just the number of values to the right of n. Therefore, we have

$$\sum_{\omega \in S_n} q^{\mathrm{inv}(\omega)} = [n]_q \sum_{u \in S_{n-1}} q^{\mathrm{inv}(u)} = [n]_q!$$

as desired. $\square$

Discovery 2.4

There must be a bijection $F : S_n \to S_n$ with

$$\mathrm{maj}(\omega) = \mathrm{inv}(F(\omega))$$

One famous such bijection is the **Foata bijection**.

2.5 Lattice Paths

Standing at the origin of $\mathbb{R}$, and only allowed to walk $\{-1, +1\}$. It is easier to look at something like the graph over time. Hence imagine we stand at the origin of $\mathbb{R}^2$, and the lattice path is a path in $\mathbb{Z}^2$ with steps $\{+1, +1\}$ and $\{+1, -1\}$ since the x-axis is representing the time (so it only increases). We name the two steps U (for "up") and D (for "down") respectively.

Lemma 2.4

We have

$$\texttt{number of lattice paths from } (0,0) \texttt{ to } (2a, 0) \texttt{ is } \binom{2a}{a}$$

Proof. We need $2a$ total steps with equal number of U's and D's, so exactly a of them are U's (or D's). $\square$

Discovery 2.5

Now we wonder how many of those never go below the x-axis? (but touching the axis is okay).

Definition 2.15: Dyck Path

These are called **Dyck paths**. The number of them is the **Catalan number** c_a.

Example 2.12

For $a = 2$, we have $c_2 = 2$.

Example 2.13

For $a = 3$, we have $c_3 = 5$.

2.6 Catalan Number

Lemma 2.5

The Catalan numbers satisfy

$$c_n = \sum_{k=0}^{n-1} c_k c_{n-k-1} \qquad \forall\, n > 0 \text{ with } c_0 = 1$$

Lecture 9 - Monday, September 23

Proof. Let

$$D_n = \{\texttt{all Dyck paths of semilength } n\}$$

We will biject D_n with $\bigcup_{k=0}^{n-1} D_k D_{n-k-1}$. For $p \in D_n$, look where it first recontacts the horizontal axis. To the right of the point of contact, the possible Dyck paths lay in D_k for some $k \in \{0, \ldots, n-1\}$. And to the left of the point of contact, the number of Dyck path lay in D_{n-1-k} instead of D_{n-k}. This is because we need the left part not to touch the horizontal axis ever, thus only Dyck path of semilength $n-1-k$ could be arbitrary (we fix the start segment and the end segment). This process yields us the recurrence relation as desired. $\square$

Result 2.5

This recurrence determines the sequence:

$$\begin{aligned}
c_0 &= 1 \\
c_1 &= c_0c_0 = 1 \\
c_2 &= c_0c_1 + c_1c_0 = 2 \\
c_3 &= c_0c_2 + c_1c_1 + c_2c_0 = 5 \\
c_4 &= c_0c_3 + c_1c_2 + c_2c_1 + c_3c_0 = 14 \\
c_5 &= c_0c_4 + c_1c_3 + c_2c_2 + c_3c_1 + c_4c_0 = 42
\end{aligned}$$

Proof. Let (from the above lemma),

$$\begin{aligned}
C(x) = \sum_{n\geq 0} c_n x^n = 1 + \sum_{n\geq 1} c_n x^n &= 1 + \sum_{n\geq 1}\left(\sum_{k=0}^{n-1} c_k c_{n-k-1}\right) x^n \\
&= 1 + \sum_{k\geq 0}\sum_{j\geq 0} c_j c_k x^{j+k+1} \\
&= 1 + x\sum_{k\geq 0}\sum_{j\geq 0} c_j c_k x^{j+k} \\
&= 1 + xC(x)^2
\end{aligned}$$

Thus

$$xC(x)^2 - C(x) + 1 = 0 \quad \Rightarrow \quad C(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$$

which is called an algebraic generating function. $\square$

2.7 Analytic Binomial Theorem

Lemma 2.6

For $\alpha \in \mathbb{C}$ and $|x| < 1$, we have

$$(1+x)^\alpha = \sum_{k\geq 0} \frac{1}{k!}(\alpha)(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)x^k$$

and we still write

$$\frac{1}{k!}(\alpha)(\alpha-1)(\alpha-2)\cdots(\alpha-k+1) = \binom{\alpha}{k}$$

Proof. Think of $(1+x)^\alpha = f$ as a function on the complex plane, $f : \mathbb{C} \to \mathbb{C}$. Some weird may happen at $x = -1$, but on the open disk $|x| < 1$, f is analytic, so f has a Taylor expansion valid on the disk and the coefficient on x^k is

$$\frac{1}{k!}\left(\frac{d}{dx}\right)^k (1+x)^\alpha\bigg|_{x=0} = \frac{1}{k!}(\alpha)(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)(1+x)^{\alpha-k}\bigg|_{x=0} = \binom{\alpha}{k}$$

$\square$

Lemma 2.7

We have

$$\sqrt{1-4x} = 1 - 2\sum_{k\geq 1}\frac{1}{k}\binom{2k-2}{k-1}x^k$$

Proof. By analytic binomial theorem,

$$\begin{aligned}\sqrt{1-4x} = (1-4x)^{1/2} &= \sum_{k\geq 0}\binom{1/2}{k}(-4k)^k \\ &= \sum_{k\geq 0}(-1)^k 4^k\binom{1/2}{k}x^k\end{aligned}$$

For $k = 0$, the coefficient for $[x^0]$ is $(-1)^0 4^0\binom{1/2}{k} = 1$ as desired. For $k \geq 1$, we rewrite

$$\begin{aligned}(-1)^k 4^k\binom{1/2}{k} &= (-1)^k 4^k\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\cdots\left(\frac{1}{2}-k+1\right)\frac{1}{k!} \\ &= (-1)4^k\frac{1}{k!}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\cdots\left(k-\frac{3}{2}\right) \\ &= -\frac{1}{k!}2^k(1)(1)(3)(5)\cdots(2k-3) \\ &= -\frac{2}{k}\frac{(1)(3)\cdots(2k-3)}{(k-1)!}2^{k-1}\cdot\frac{(k-1)!}{(k-1)!} \\ &= -\frac{2}{k}\cdot\frac{1\cdot 3\cdot 5\cdots(2k-3)}{(k-1)!}\cdot\frac{2\cdot 4\cdot 6\cdots(2k-2)}{(k-1)!} \\ &= -\frac{2}{k}\cdot\frac{(2k-2)!}{(k-1)!(k-1)!} \\ &= -\frac{2}{k}\binom{2k-2}{k-1}\end{aligned}$$

as desired. ☐

Lecture 10 - Wednesday, September 25

Theorem 2.9

We have

$$c_n = \frac{1}{n+1}\binom{2n}{n}$$

Proof. We have

$$\begin{aligned}c_n = [x^n]C(x) &= [x^n]\frac{1\pm\sqrt{1-4x}}{2x} \\ &= [x^n]\left(\frac{1}{2x}\pm\frac{1}{2x}\left(1-2\sum_{k\geq 1}\frac{1}{k}\binom{2k-2}{k-1}x^k\right)\right)\end{aligned}$$

Notice that $C(x)$ must be using the "$-$" sign in its expression because otherwise it is not a FPS and has negative coefficients.

Definition 2.16: Laurent Series

The above seires is known as a **Laurent Series**.

Therefore, we have

$$\begin{aligned} c_n &= [x^n] \sum_{k \geq 1} \frac{1}{k} \binom{2k-2}{k-1} x^{k-1} \\ &= [x^n] \sum_{n \geq 0} \frac{1}{n+1} \binom{2n}{n} x^n = \frac{1}{n+1} \binom{2n}{n} \end{aligned}$$

as desired. □

2.7.1 More direct proof of the general form of $c_n = \dfrac{1}{n+1}\dbinom{2n}{n}$

For $a, b \in \mathbb{Z}^2$, let $P(a \to b)$ be the set of lattice paths from a to b. We already computed

$$\big| P((0,0) \to (2n,0)) \big| = \binom{2n}{n}$$

Hence to determine $c_n = |D_n|$, it suffices to count

$$P((0,0) \to (2n,0)) \backslash D_n := B_n \qquad \text{"}\textbf{B}\text{ad paths"}$$

Given $b \in B_n$, it eventually meets the line $y = -1$. At the first position where it meets, take the subsequent segments of b and reflect them across $y = -1$, call them $\text{refl}(b)$. Notice that now the desitination ends up being the point $(2n, -2)$. What kind of lattice path is this?

Note that the first part of the original path has one more D than U, and the second part, $\text{refl}(b)$, also has one more D than U. Hence it is a path in

$$P((0,0) \to (2n,-2)) := R_n$$

Observe that given $r \in R_n$, it also meets $y = -1$ at some point (sort of the discrete intermediate value theorem). At the first such meeting, reflect the subsequent segments across the line $y = -1$. We now get a path, denoted as $\text{refl}(r) \in B_n$. Easy to see that

$$\text{refl}^2 = \text{id}$$

Hence

$$\text{refl} : B_n \leftrightarrow R_n$$

is a bijection. But

$$|R_n| = \binom{2n}{n-1} = \frac{2n!}{(n-1)!(n+1)!} = \frac{2n!}{n!n!} \frac{n}{n+1} = \binom{2n}{n} \frac{n}{n+1}$$

Thus

$$|D_n| = \binom{2n}{n} - \binom{2n}{n}\frac{n}{n+1} = \frac{1}{n+1}\binom{2n}{n}$$

Lecture 11 - Friday, September 27

2.8 More Catalan Objects

Example 2.14

Each Dyck path of semilength n can be converted into a string of length $2n$, whose entries are filled with U's and D's representing "up" and "down" respectively. In addition to that, the number of U and D are the same, and every initial segment has at least as many U as D.
As an analogue, we may also view U's and D's as open and close parentheses that pair sensibly.

3 Linear Recurrences

Let $A(x) = \dfrac{2+5x}{1-3x^2-2x^3} = \sum_{n\geq 0} a_n x^n = \sum_{x\in\mathbb{Z}} a_n x^n$ with $a_k = 0$ for $k < 0$. Therefore, we have

$$(1-3x^2-2x^3)A(x) = 2+5x =: p(x)$$

We notice that

$$\begin{aligned} [x^m]p(x) &= \begin{cases} 2 & \text{if } m = 0 \\ 5 & \text{if } m = 1 \\ 0 & \text{if } m > 1 \end{cases} \\ &= [x^m](1-3x^2-2x^3)A(x) \\ &= [x^m]\left(A(x) - 3x^2A(x) - 2x^3A(x)\right) \\ &= [x^m]A(x) - 3[x^{m-2}]A(x) - 2[x^{m-3}]A(x) \\ &= a_m - 3a_{m-2} - 2a_{m-3} \end{aligned}$$

Notice that we have

m	
0	$a_0 = 2$
1	$a_1 = 5$
2	$a_2 - 3a_0 = 0 \quad \Rightarrow \quad a_2 = 6$

Notice that for $m > 1$, we have the linear recurrence

$$a_m = 3a_{m-2} + 2a_{m-3}$$

Discovery 3.1

Notice that the coefficients match those are in the denominator.

In general, suppose $A(x) = \sum_{n\geq 0} a_n x^n = \dfrac{p(x)}{q(x)}$ with p, q are polynomials and $q_0 = 1$. Say

$$p(x) = \sum_{i=0}^{d} p_i x^i \qquad \text{and} \qquad q(x) = \sum_{j=0}^{e} q_j x^j$$

then we have

$$q(x)A(x) = p(x)$$

We may also write down their coefficients:

$$
\begin{aligned}
[x^m]p(x) &= \begin{cases} p_m & \text{if } x \leq d \\ 0 & \text{if } m > d \end{cases} \\
&= [x^m]q(x)A(x) \\
&= [x^m]\left(\sum_{j=0}^{e} q_i x^i\right) A(x) \\
&= \sum_{j=0}^{e} q_i [x^m] x^i A(x) \\
&= \sum_{j=0}^{e} q_i [x^{m-i}] A(x) \\
&= \sum_{j=0}^{e} q_i a_{m-i}
\end{aligned}
$$

Hence for $m > d$, we have

$$\sum_{i=0}^{e} q_i a_{m-i} = 0 \quad \Rightarrow \quad \underbrace{q_0}_{=1} a_m = -\sum_{i=1}^{e} q_i a_{m-1}$$

Moreover, for $m \leq d$, we get initial conditions (recall that $q_0 = 0$)

$$
\begin{array}{c|rcl}
m = 0 & q_0 a_0 = p_0 & \Rightarrow & a_0 = p_0 \\
m = 1 & q_1 a_0 + q_0 a_1 = p_1 & \Rightarrow & a_1 = p_1 - q_1 a_0 \\
m = 2 & q_2 a_0 + q_1 a_1 + q_0 a_2 = p_2 & \Rightarrow & a_2 = p_2 - q_1 a_1 - q_2 a_0 \\
\vdots & & & \\
m = d & & a_d = p_d - a_0 q_d - a_1 q_{d-1} - \cdots - a_{d-1} q_1 &
\end{array}
$$

Theorem 3.1

Let $g_0, g_1, g_2, \ldots$ be a sequence of numbers, not necessarily positive integers. Let

$$G(x) = \sum_{k \geq 0} g_k x^k$$

be the generating series. TFAE:

1. The sequence satisfies a homogenous linear recurrence relation

$$g_n + a_1 g_{n-1} + \cdots + a_d g_{n-d} = 0$$

   for all $n >$ some N. With initial conditions $g_0, g_1, \ldots, g_N$.

2. $G(x) = \dfrac{p(x)}{q(x)}$ is a quotient of 2 polynomials. Moreover, $q(x) = 1 + a_1 x + a_2 x^2 + \cdots a_d x^d$, and $p(x) = p_0 + p_1 x + \cdots + p_N x^N$ where $p_k = g_k + a_1 g_{k-1} + \cdots + a_d g_{k-d}$.

Example 3.1

Suppose we have

$$A(x) = \frac{1 + x + x^2}{1 + 3x + 2x^2 + x^4} = \sum a_n x^n$$

Our recurrence would be

$$a_n = -3a_{n-1} - 2a_{n-2} - 1a_{n-4}$$

where $a_0 = p_0 = 1$, $a_1 = p_1 - q_1 a_0 = 1 - 3 = -2$, $a_2 = p_2 - q_1 a_1 - q_2 a_0 = 1 + 6 - 2 = 5$.

Lecture 12 - Monday, September 30

3.1 Extracting Coefficients (for rational generating series)

If the generating function is in the form of $\frac{?}{1 - ax}$, then we can easily handle it since it is a geometric series. Likewise, if it is in the form of $\frac{?}{(1 - bx)^k}$, then we can still easily handle it using the negative binomial theorem.

Example 3.2

Suppose we have

$$A(x) = \frac{3 + x}{1 - x}$$

If we want to extract the coefficient of the term x^{100}, then

$$\begin{aligned}
[x^{100}]A(x) &= [x^{100}]\frac{3}{1 - x} + [x^{100}]\frac{x}{1 - x} \\
&= 3[x^{100}]\frac{1}{1 - x} + [x^{99}]\frac{1}{1 - x} \\
&= 3 + 1 = 4
\end{aligned}$$

In general, we have

$$[x^n]A(x) = 4 \qquad \forall\, n \geq 1$$

Example 3.3

Suppose we have

$$B(x) = \frac{x^2}{1-2x}$$

If we want to extract the coefficient of the term x^{100}, then

$$[x^{100}]B(x) = [x^{100}]\frac{x^2}{1-2x} = [x^{98}]\frac{1}{1-2x} = 2^{98}$$

In general, we have

$$[x^n]B(x) = 2^{n-2} \qquad \forall\, n \geq 2$$

Example 3.4

Suppose we have

$$C(x) = \frac{1+x}{(1-x)^3}$$

If we want to extract the coefficient of the term x^{100}, then

$$\begin{aligned}[x^{100}]C(x) &= [x^{100}]\frac{1}{(1-x)^3} + [x^{100}]\frac{x}{(1-x)^3} \\ &= \binom{100+3-1}{3-1} + \binom{99+3-1}{3-1} \\ &= \binom{102}{2} + \binom{101}{2}\end{aligned}$$

In general, we have

$$[x^n]C(x) = \binom{n+2}{2} + \binom{n+1}{2} \qquad \forall\, n \geq 1$$

Example 3.5

Suppose we have

$$D(x) = \frac{1}{(1-2x)^5}$$

If we want to extract the coefficient of the term x^{100}, then

$$\begin{aligned}[x^{100}]D(x) &= [x^{100}]\frac{1}{(1-2x)^5} \\ &= \binom{100+5-1}{5-1}2^{100} = \binom{104}{4}2^{100}\end{aligned}$$

In general, we have

$$[x^n]D(x) = \binom{n+4}{4}2^n \qquad \forall\, n \geq 0$$

3.1.1 More Complicated Denominators

Example 3.6

Suppose we have

$$E(x) = \frac{1+x}{1-5x+6x^2}$$

Suppose we want to extract the coefficient for the term x^n. We have

$$\begin{aligned}[x^n]E(x) &= [x^n]\frac{1+x}{(1-2x)(1-3x)} \\ &= [x^n]\frac{-3}{1-2x} + [x^n]\frac{4}{1-3x}\end{aligned}$$

which is in the form we know how to solve.

Discovery 3.2

If we have a rational function such that deg(numerator) $\geq$ deg(denominator), then we need to simplify it using long division.

3.1.2 Partial Fractions Theorem, (first version)

Theorem 3.2: Partial Fractions, first version

Suppose $\deg(P(x)) < \deg(Q(x)) = d$ and

$$\frac{P(x)}{Q(x)} = \frac{P(x)}{(1-\lambda_1 x)\cdots(1-\lambda_d x)}$$

with all $\lambda_1, \ldots, \lambda_d \in \mathbb{C}$ distinct. Then there exist $c_1, \ldots, c_d \in \mathbb{C}$ such that

$$\frac{P(x)}{Q(x)} = \frac{c_1}{1-\lambda_1 x} + \cdots + \frac{c_d}{1-\lambda_d x}$$

Proof. How to find the c_k's?
We need to put everything together, on top of a common denominator (this involving cross-multiply). $\square$

3.1.3 Partial Fractions Theorem, (full version)

Theorem 3.3: Partial Fractions, full version

Suppose $\deg(P(x)) < \deg(Q(x)) = d$ and suppose Q factors as

$$Q(x) = (1 - \lambda_1 x)^{m_1} \cdots (1 - \lambda_s x)^{m_s}$$

where $m_i \geq 1$. $\sum m_i = d$, and all $\lambda_i \in \mathbb{C}$ are distinct. Then there exists

$$c_{11}, \ldots, c_{1m_1}, \ldots, c_{s1}, \ldots, c_{sm_s}$$

such that

$$\frac{P(x)}{Q(x)} = \sum_{i=1}^{s} \sum_{j=1}^{m_i} \frac{c_{ij}}{(1 - \lambda_i x)^j}$$

Lecture 13 - Wednesday, October 02

Example 3.7

Suppose we have

$$A(x) = \frac{-2 + 6x + 2x^2}{(1 - 2x)^2(1 - 5x)}$$

By the above theorem, we know that we can partial fraction $A(x)$ in the form of

$$A(x) = \frac{c_{11}}{1 - 2x} + \frac{c_{12}}{(1 - 2x)^2} + \frac{c_{21}}{1 - 5x}$$

Cross-multiply yields

$$-2 + 6x + 2x^2 = c_{11}(1 - 2x)(1 - 5x) + c_{12}(1 - 5x) + c_{21}(1 - 2x)^2 \tag{1}$$

thus we have

$$\begin{array}{rrrcrcrcr} c_{11} & +c_{12} & +c_{21} & = & -2 & & c_{11} & = & 1 \\ -7c_{11} & -5c_{12} & -4c_{21} & = & 6 & \Rightarrow & c_{12} & = & -1 \\ 10c_{11} & & +4c_{21} & = & 2 & & c_{21} & = & -2 \end{array}$$

Discovery 3.3

Here is a trick for solving such problems:

`evaluate the equation at clever values of` x

For an instance, we may evaluate the above equation(1) at $x = 1/2$ or $x = 1/5$.

Example 3.8

Now we have

$$\begin{aligned}
A(x) &= \frac{1}{1-2x} + \frac{-1}{(1-2x)^2} + \frac{-2}{1-5x} \\
&= \sum_{n\geq 0} (2x)^n - \binom{n+2-1}{2-1}(2x)^n - 2\cdot(5x)^n \\
&= \sum_{n\geq 0} \left(2^n - (n+1)\cdot 2^n - 2\cdot 5^n\right) x^n \\
&= \sum_{n\geq 0} \left(-2\cdot 5^n - n\cdot 2^n\right) x^n
\end{aligned}$$

3.1.4 Factoring Denominator

Example 3.9

Suppose we want to factor

$$1-2x-3x^2 = (1-3x)(1+x)$$

This is different from the convention we used to have in high school, so here is a trick for solving it:

$$\begin{aligned}
1-2x-3x^2 = (1-3x)(1+x) &= x^2\left(\left(\frac{1}{x}\right)^2 - 2\frac{1}{x} - 3\right) \\
&= x^2(y^2-2y-3) \\
&= x^2(y-3)(y+1) = (1-3x)(1+x)
\end{aligned}$$

3.1.5 General Expression of Fibonacci

Example 3.10

We want to factor

$$f_0 = 0,\ f_1 = 1,\ f_2 = 1 \qquad f_n = f_{n-1} + f_{n-2} \quad \forall\, n \geq 2$$

which is the Fibonacci sequence. We have generating function

$$F(x) = \sum_{n\geq 0} f_n x^n = \frac{x}{1-x-x^2}$$

We wonder if we can get a general formula for $f_n = [x^n]F(x)$. Notice that we have

$$1-x-x^2 = \left(1 - \frac{1+\sqrt{5}}{2}x\right)\left(1 - \frac{1-\sqrt{5}}{2}x\right)$$

Thus we have

$$\frac{x}{1-x-x^2} = \frac{A}{1-\frac{1+\sqrt{5}}{2}x} + \frac{B}{1-\frac{1-\sqrt{5}}{2}x}$$

Cross-multiply we get

$$A = \frac{1}{\sqrt{5}} \qquad \text{and} \qquad B = \frac{-1}{\sqrt{5}}$$

Now we have

$$\begin{aligned}
\frac{x}{1-x-x^2} &= \frac{1}{\sqrt{5}}\left(\frac{1}{1-\frac{1+\sqrt{5}}{2}x}\right) - \frac{1}{\sqrt{5}}\left(\frac{1}{1-\frac{1-\sqrt{5}}{2}x}\right) \\
&= \frac{1}{\sqrt{5}}\sum_{n\geq 1}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)x^n \\
\Rightarrow \quad f_n &= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}
\end{aligned}$$

Hence we have the approximation: For $n \gg 0$, we have

$$f_n \approx \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n$$

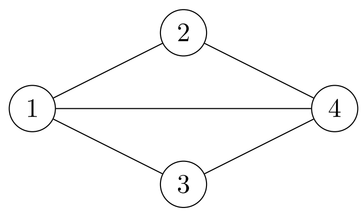
Lecture 14 - Friday, October 04

4 Graphs

Definition 4.1: Graph

A **graph** is a pair $G = (V, E)$ where V is a set of **vertices** and $E \subseteq \binom{V}{2}$ is the set of **edges**.

Example 4.1



Thus we have

$$V = \{1, 2, 3, 4\}$$
$$E = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 1\}, \{1, 3\}\}$$

Definition 4.2: Incident

$\{1, 2\}$ and $\{1, 3\}$ are **incident** to each other at the vertex 1. They are also **incident** to vertex 1.

Definition 4.3: Adjacent/ Joined

We say vertices 2 and 4 are **adjacent**/ or **joined** because there is an edge connecting the two vertices. We also say that they are **neighbours**.

Definition 4.4: Degree

The degree $\deg(v)$ of vertex v is the number of neighbours v has, or the number of edges incident at v.

Definition 4.5: Vertex Set and Edge Set

If G is a graph, we write $V(G)$ for the vertex set, and $E(G)$ for the edge set.

4.1 Handshake Theorem

Theorem 4.1: Handshake Theorem

For any graph G,

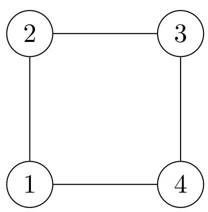
$$\sum_{v \in V(G)} \deg(v) = 2|E(G)|$$

Proof. Let $A = \{(v, e) : v \in V(G), e \in E(G), v \in e\}$, then

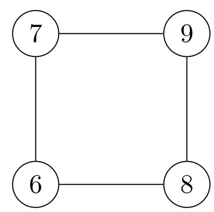
$$|A| = \sum_{v \in V(G)} \deg(v) = 2|E(G)|$$

as desired. $\square$

Example 4.2: Isomorphic Graphs



Graph G.



Graph H.

We notice that $G \neq H$ because they are built out of different vertices, but G and H are isomorphic to each other, denoted as

$$G \cong H$$

Informally, two graphs are isomorphic if we can make them look like the same picture except for the vertex labels. Formally,

$$f : V(K) \longrightarrow V(K')$$

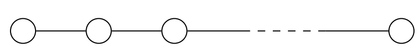
that respect adjacency, i.e.,

$$E(K) \ni \{u, v\} \quad \longleftrightarrow \quad \{f(u), f(v)\} \in E(K')$$

4.2 List of Graphs

Definition 4.6: Path Graph

The **path graph** P_n has n vertices and $n - 1$ edges and looks like



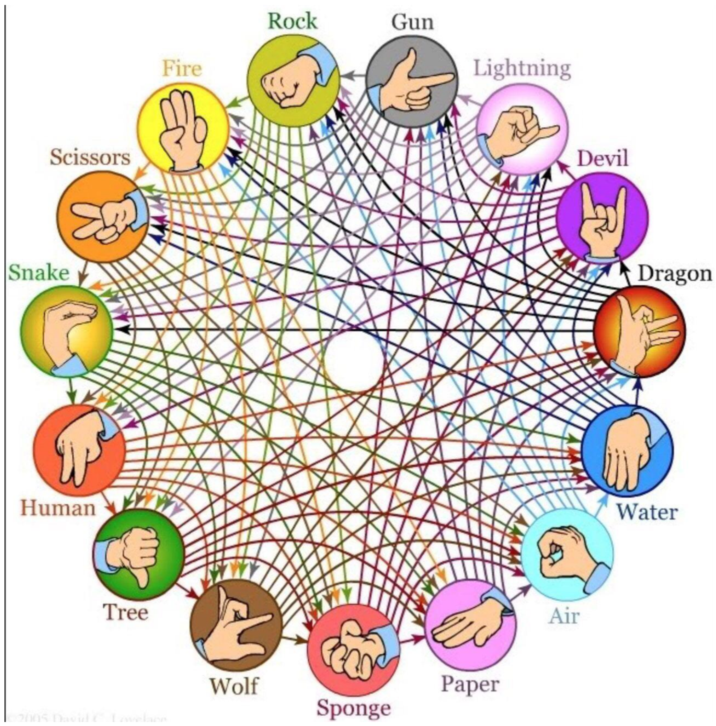
Definition 4.7: Cycle

The **cycle** C_n has n vertices and n edges and looks like "a circle". Note that a triangle or a square is also a "circle".

Definition 4.8: Complete Graph

The **complete graph** K_n has n vertices and $E(K_n) = \binom{V(K_n)}{2}$.

Example 4.3: Complete graph



Definition 4.9: Edgeless Graph

The **edgeless graph** E_n has n vertices and no edges.

Definition 4.10: Empty Graph

The **empty graph** has no vertices and no edges. The picture looks like:

Definition 4.11: Hypercube

The **hypercube** Q_n has $V(Q_n) = \{0,1\}^n$ with two strings adjacent if they differ in exactly one coordinate.

Definition 4.12: Kneser Graph

The **Kneser graph** $K(n,m)$ has vertex set $V(K(n,m)) = \binom{[n]}{m}$ with $\{u,v\} \in E(K(n,m))$ if $u \cap v = \varnothing$.

Definition 4.13: Petersen Graph

Basically $K(5,2)$.

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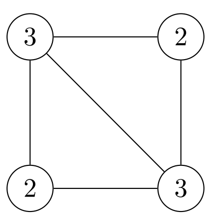
Definition 4.14: Hadwiger-Nelson

The **Hadwiger-Nelson graph** G has $V(G) = \mathbb{R}^2$ and $\{u,v\} \in E(G)$ if and only if $|u-v| = 1$.

Definition 4.15: Degree Multiset

Every graph G has a **degree multiset** $\{\{\deg(v) : v \in V(G)\}\}$.
Remark: people don't like multisets, so usually we sort it to the degree sequence (weakly decreasing).

Example 4.4



Graph H.

We have the degree multiset as

$$\{\{2,3,2,3\}\}$$

hence the degree sequence is 3322.

Definition 4.16: K-regular

A graph G is k**-regular** if all vertices have degree k. G is **regular** if it is k-regular for some k.

Discovery 4.1

We discover that complete graphs K_n are $(n-1)$-regular, and hypercubes Q_n are n-regular. Moreover. Kneser graphs are $\binom{n-m}{m}$-regular.

How to tell if 2 graphs are isomorphic?

4.2.1 Graph Invariant

Algorithm 4.1

If two graphs are isomorphic, we try to find an isomorphism. If they are not, we find a graph invariant to distinguish them.

Definition 4.17: Graph Invariant

A **graph invariant** is a property of graphs such that G, H have the same value of the property whenever $G \cong H$.

Example 4.5: Example of Invariant

1. $|V(G)|$
2. $|E(G)|$

Lemma 4.1

The degree sequence is also a graph invariant.

Proof. Suppose $f : G \to H$ is an isomorphism, then for each $v \in V(G)$, f maps the neighbours of v to the neighbours of $f(v)$ bijectively. Hence

$$\deg(v) = \deg(f(v))$$

Thus the degree multiset and the degree sequence does not change. $\square$

4.3 Remark on Complexity of Graph Isomorphism

We do not have a polynomial time algorithm, but maybe there is one. It might be an NP-complete but probably not.

So far, we have an algorithm with $2^{O((\log n)^3)}$.

Definition 4.18: Automorphism

An **automorphism** of G is an isomorphism $\varphi : G \to G$. We write $\mathrm{Aut}(G)$ for the set of automorphisms and we also define $\mathrm{aut}(G) = |\mathrm{Aut}(G)|$.

Lemma 4.2

$|\mathrm{Aut}(G)|$ is a graph invariant.

Proof. Suppose $G \cong H$.

$$G \overset{f}{\leftrightharpoons} H$$

Then $\alpha : \mathrm{Aut}(G) \to \mathrm{Aut}(H)$ defined as $\varphi \overset{\alpha}{\mapsto} f \circ \varphi \circ f^{-1}$ is an injection. But also, $\beta : \mathrm{Aut}(H) \to \mathrm{Aut}(G)$ defined as $\psi \overset{\beta}{\mapsto} f^{-1} \circ \varphi \circ f$ is also an injection. Hence we have $|\mathrm{Aut}(G)| = |\mathrm{Aut}(H)|$. $\square$

Discovery 4.2

Actually, $\mathrm{Aut}(G)$ is a group, and this group is also a group invariant.

Result 4.1

It is easy to see that

$$\mathrm{aut}(Q_n) = 2^n \cdot n!$$

We may choose the binary string $(0 \ldots 0)$ any where among all the 2^n choices, and then we permutate the n neightbours. The number of neighbours is $n!$ because we may alter any one of the n components in the binary string of length n.

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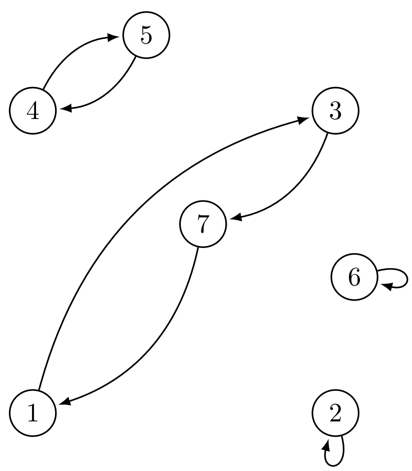
Definition 4.19: Directed Graph

A **directed graph** (**digraph**) is a pair $D(V, E)$ where V is a set of vertices and $E \subseteq V \times V$ is a set of edges.

Example 4.6

For instance $V = [7]$ and $E = \{(1,3),(3,7),(2,2),(7,1),(6,6),(4,5),(5,4)\}$



Definition 4.20: Head and Tail

If (i, j) is an edge of a directed graph D, we call j the **head** and i the **tail**.

Definition 4.21: Indegree

A vertex $v \in V(D)$ has an **indegree**

$$\text{indeg}(v) = \left|\{u \in V(D) : (u, v) \in E(D)\}\right|$$

Similarly, we have an analogous definition for outdeg as well.

Definition 4.22: Functional Digraph

Every permuatation $\omega \in S_n$ has an associated digraph. The **functional digraph** D_ω is a directed graph with $V(D_\omega) = [n]$ and $E(D_\omega) = \{(k, \omega(k)) : k \in [n]\}$.

Discovery 4.3

Because ω is a bijection, thus for all $v \in V(D_\omega)$, we have

$$\text{indeg}(v) = \text{outdeg}(v) = 1$$

Moreover, D_ω is a disjoint union of some *oriented* cycles (heads only connect to tails).

Result 4.2

D_ω is another notation to specify ω.

4.3.1 Weight Function : cyc

Definition 4.23:

We also denote the number of cycles in ω as $\text{cyc}(\omega)$.

4.4 Stirling Numbers of the First Kind

Now we have the generating series

$$\sum_{\omega \in S_n} x^{\text{cyc}(\omega)} = \sum_{k=0}^{n} c(n, k) x^k$$

where the coefficient $c(n, k)$ are *signless Stirling numbers of the first kind.* The actual Stirling numbers of the first kind are

$$(-1)^{n-k} c(n, k)$$

Proposition 4.1

We have

$$c(n,k) = c(n-1,k-1) + (n-1)\, c(n-1,k)$$

Proof. Let $S(n,k)$ be the set of permutations of n with exactly k cycles. We wish to biject

$$S(n,k) \longleftrightarrow S(n-1,k-1) \sqcup S(n-1,k) \times [n-1]$$

Our map goes from the right hand side to the left hand side. Given $\omega \in S(n-1,k-1)$, define $f(\omega) \in S(n,k)$ by

$$f(\omega)(k) = \begin{cases} \omega(k) & \text{if } k < n \\ n & \text{if } k = n \end{cases}$$

Given $(\omega, i) \in S(n-1,k) \times [n-1]$, define $f(\omega, i) \in S(n,k)$ by

$$f(\omega,i)(k) = \begin{cases} n & \text{if } k = i \\ \omega(i) & \text{if } k = n \\ \omega(k) & \text{otherwise} \end{cases}$$

Easy to check (**exercise**) this is a bijection. $\square$

4.4.1 Stirling's First Kinds $\triangle$

n = 0						1					
n = 1					0		1				
n = 2				0		1		1			
n = 3			0		2		3		1		
n = 4		0		6		11		6		1	
n = 5	0		24		50		35		10		1

Discovery 4.4

Observe the pattern for the red-circled sequence:

$$a_n = (n-1)!$$

and the pattern for the blue-circled sequence:

$$b_n = \binom{n}{2}$$

Moreover, the sum of all the entries on each row n is $n!$, the total number of permuatation of S_n.

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Theorem 4.2

We have

$$\sum_{\omega \in S_n} x^{\mathrm{cyc}(\omega)} = \sum_{k=0}^{n} c(n,k)x^k = x(x+1)\cdots(x+n-1)$$

Proof. Let $F_n(x) = x(x+1)\cdots(x+n-1) = \sum_{k=0}^{n} f(n,k)x^k$. We wish to show that $c(n,k) = f(n,k)$. We have $f(0,0) = 1$ and $f(n,k) = 0$ if $n < 0$ or $k < 0$. Now

$$\begin{aligned} F_n(x) &= (x+n-1)F_{n-1}(x) = xF_{n-1}(x) + (n-1)F_{n-1}(x) \\ &= \sum_{k=1}^{n} f(n-1,k-1)x^k + \sum_{k=0}^{n-1} f(n-1,k)x^k \end{aligned}$$

Thus

$$f(n,k) = f(n-1,k-1) + (n-1)f(n-1,k)$$

which has the same initial condition and the same recurrence relation as $c(n,k)$, hence they are the same. $\square$

4.5 Cycle Notation of Permutations

Definition 4.24: Cycle Notation

A **cycle notation** of a permuatation $\omega \in S_n$ is given as follows: Decompose D_ω (digrah) into a disjoint union of oriented cycles. Write down each cycle in order (starting somewhere) between parentheses and then concatenate all the cycles.

Example 4.7

Consider the same example as in Example (4.6), hence we may write

$$\omega = (137)(6)(54)(2) = (6)(713)(2)(45) = \cdots$$

Definition 4.25: Standard Cycle Notation

Standard cycle notation is to

1. write each cycle starting with its greatest element, and
2. list the cycles in increasing order of their greatest element.

Example 4.8

Consider the same example as in Example (4.6) again, we have

$$\omega = (2)(54)(6)(713)$$

Definition 4.26:

Define $\zeta : S_n \to S_n$ to be the map that delete the parentheses and then treats the result as one-line notation.

Lemma 4.3

We can recover ω from $\zeta(\omega)$ by inserting '(' before each Left to Right maximum and inserting ')' in the appropriate places, so ζ is a bijection.

Proof. Check this. ☐

4.5.1 Weight Function: l2r

Discovery 4.5

Note that ω has k cycles if and only if $\zeta(\omega)$ has k Left to Right maxima. Let $\text{l2r}(\omega)$ to be the number of left to right maxima.

Corollary 4.1

We have

$$\sum_{\omega \in S_n} x^{\text{l2r}(\omega)} = \sum_{\omega \in S_n} x^{\text{cyc}(\omega)} = \sum_{k=0}^{n} c(n,k)x^k = x(x+1)\cdots(x+n-1)$$

4.6 Back to Graphs

Definition 4.27: Subgraph

A graoh $H = (V', E')$ is a **subgraph** of a graph $G(V, E)$ if $V' \subseteq V$ and $E' \subseteq E$.

Discovery 4.6

Not every (V'', E'') with $V'' \subseteq V$ and $E'' \subseteq E$ is a subgraph, it has to be a graph itself first.

Definition 4.28: Spanning

A subgraph H of G is called **spanning** if $V(H) = V(G)$.

Definition 4.29: Induced

A subgraph H of G is **induced** if

$$E(H) = \{e \in E(G) : e \subseteq V(H)\}$$

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Definition 4.30: Connected

G is **connected** if it is non-empty and there is a path between every pair of vertices.

Definition 4.31: Connected Component

A **connected component** is a maximal connected subgraph.

Lemma 4.4

Every graph G decomposes uniquely as a disjoint union of connected graphs.

5 Forest, Trees and Leaves

Definition 5.1: Forest and Tree and Leaf

G is a **forest** if it has no cycle. G is a **tree** if it is a connected forest. A vertex v in a tree is a **leaf** if $\deg(v) = 1$.

Proposition 5.1

Every finite tree, say T, with at least 2 vertices has at least two leaves.

Proof. Consider a longest path in T, say it starts at x and end at y. Then $x \neq y$ because it has at least two vertices. Then we have

$$\deg(x) = \deg(y) = 1$$

This is because if x has another neightbour, then the neighbour is either a vertex on the path, or a vertex somewhere else. If the neighbour is on the path, we have a cycle, which cannot be true; else if the neighbour is a vertex somewhere else, the path is no longer the longest path, which cannot be true either. $\square$

Proposition 5.2

Every connected graph has a spanning subtree.

Proof. If G isn't a tree, find a cycle in G and delete any edge in the cycle. The graph is still connected. Repeat deleting edges until it's a tree. $\square$

Proposition 5.3

An n-vertex tree T has $n - 1$ edges.

Proof. Delete edges one by one. At each step, number of connected components increase by one. We start with 1 component and end with n components. Hence we we deleted $n - 1$ edges. $\square$

Corollary 5.1

An n-vertex forest with k components has $n - k$ edges.

5.1 How many trees T with $V(T) = [n]$?

Theorem 5.1: Cayley

Number of trees on vertex set $[n]$ is n^{n-2}.

Proof. Let T_n be the set of trees on $[n]$. Consider E_n the set of functions $f : [n] \to [n]$ with $f(1) = 1$ and $f(n) = n$. Such a function f has a "two-line notation"

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & \cdots & n \\ f(1) & f(2) & f(3) & f(4) & \cdots & f(n) \end{array}$$

and a "one-line notation"

$$f(2)f(3)f(4)\cdots f(n-1)$$

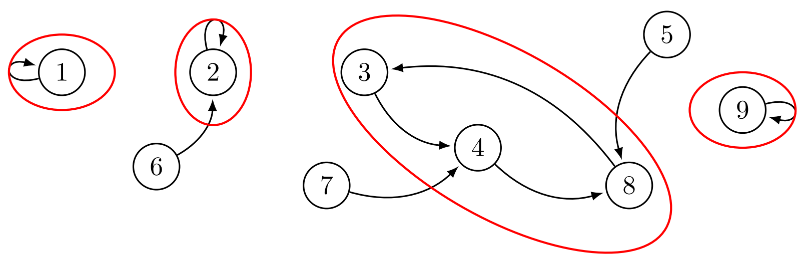
So $|E_n| = n^{n-2}$ (each of the $f(i)$'s has n choices, and there are $n-2$ of them).
Now we wish to exhibit a bijection between E_n and T_n. Look at the functional digraph of $f \in E_n$ where directed edges $i \to j$ if $f(i) = j$.

Example 5.1

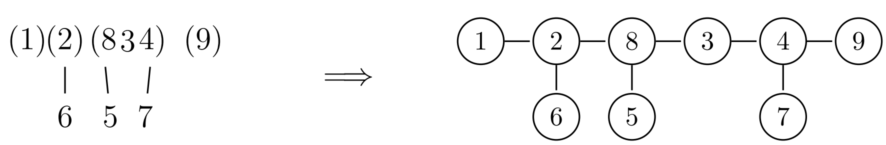
For instance, suppose we have in E_n shown as following:

$$\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 2 & 4 & 8 & 8 & 2 & 4 & 3 & 9 \end{array}$$

Hence the digraph would look like:



Now we consider the cycles in the diagraph and write down the standard cycle notation for them, add the dangling trees, delete the parentheses, and then connect the "spine" to create the tree:



$\square$

5.2 Hamiltonian Cycle

Definition 5.2: Hamiltonian Cycle

A **spanning cycle/ path** in a graph G is called a **Hamiltonian Cycle/ path**

Comment 5.1

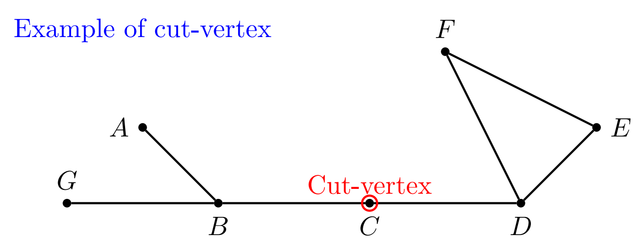
Which graphs have a Hamiltonian Cycle???

Discovery 5.1

Here are some necessary conditions:

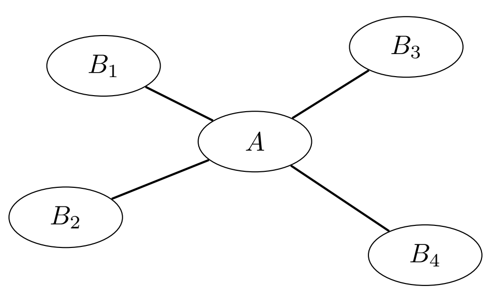
- connected;
- every vertex must be in a cycle;
- no **cut-vertex** (a vertex that cuts the graph into two if is deleted);

Example 5.2



- cannot have blobs where |centre blob| < number of outside blobs;

Example 5.3



In this example, we cannot have $|A| < 4$.

However, these are not sufficient, an counterexample would be the Petersen graph.

5.2.1 Testing if a graph has a H-cycle is NP-complete.

Theorem 5.2

Testing if a graph has a H-cycle is NP-complete.

Definition 5.3: Vertex-transitive

A graph G is **vertex-transitive** if for all $u, v \in V(G)$, there is a $\varphi \in \text{Aut}(G)$ with $\varphi(u) = v$.

Example 5.4

Vertex-transitive graphs are k-regular. As an example, Kneser graphs are vertex transitive. Moreover, cycles and complete graphs are also.

Example 5.5

Trees, T, are almost never vertex transitive, unless $|T| < 3$.

5.3 Conjecture: Lovasz 1969

Every finite connected vertex transitive graph has a H-path.

5.4 Conjecture: Babai 1996

There are lots of finite connected vertex transitive graphs with NO H-path.

Comment 5.2

Notice that it cannot be the case that both conjectures are true.

5.4.1 More Conjecture

There are exactly 5 finite connected vertex transitive graphs without H-cycle (K_2, Petersen, and three more).

Comment 5.3

Do people care about K_1?

5.5 Theorems

5.5.1 Merino, Mutze, Namrata 2023

Theorem 5.3: Merino, Mutze, Namrata 2023

Every connected Kneser graph has a H-cycle except $K(5,2)$.

5.5.2 Dirac 1952 (sufficient conditions for Hamiltonian cycle)

Theorem 5.4: Dirac 1952

If G is a graph with $n \geq 3$ vertices and all degrees are at least $n/2$, then G has a Hamiltonian cycle.

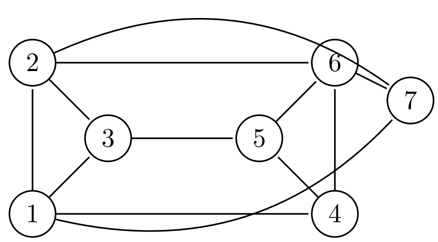
Proof. The proof follows from the next theorem. □

5.5.3 Ore 1960

Theorem 5.5: Ore 1960

If G is a graph with $n \geq 3$ vertices such that for all $u \neq v \in V(G)$ with $\{u, v\} \notin E(G)$, we have $\deg(u) + \deg(v) \geq n$, then G has a Hamiltonian cycle.

Example 5.6



Notice we have the cycle: 2135467.

Proof. Consider a counterexample and saturate it: adding edges as long as we can without creating a H-cycle. Let G be the result, then G has no H-cycle but adding an edge makes a H-cycle. Look at $x \neq y \in V(G)$ with $\{x, y\} \notin E(G)$. Then by saturatedness, there is a Hamiltonian path from x to y:

$$x = x_1 \;—\; x_2 \;—\; x_3 \;—\; \cdots \;—\; x_n = y$$

Observe if $\{x, x_i\} \in E(G)$, then $\{x_{i-1}, y\} \notin E(G)$ because otherwise

$$x_1 x_2 \cdots x_{i-1} y x_n x_{n-1} \cdots x_i x_1$$

is a H-cycle. Hence

$$\deg(y) \leq n - 2 - (\deg(x) - 1) = n - 1 - \deg(x) \quad \Rightarrow \quad \deg(x) + \deg(y) \leq n - 1$$

which is a contradiction. $\square$

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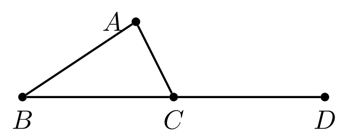
Definition 5.4: Walk

A **walk** in a graph G is a sequence of vertices $v_0 v_1 \ldots v_k$ such that such $\{v_i, v_{i+1}\} \in E(G)$. The length of a walk is k, the number of edges traversed. If $v_0 = v_k$, the walk is **closed**.

Example 5.7



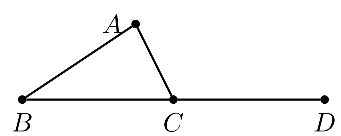
An example of a walk would be

$$ABCBCBABCDCDCA$$

Definition 5.5: Trail

A **trail** is a walk where edges never repeat.

Example 5.8



An example of a walk would be $CABCD$.

5.5.4 Euler Circuit

Definition 5.6: Circuit

A closed trail is called a **circuit**.

Comment 5.4

An Euler Circuit is a circuit that uses every edge exactly once.

Definition 5.7: Path

A path is a walk that does not repeat **vertices**.

Discovery 5.2

Every path is a trail.

Theorem 5.6: Euler 1736, Hierholzer 1873

A connected graph G has an Euler circuit if and only if every vertices has even degrees.

Proof. `Forward direction`:
The condition is necessary because we have to leave each vertex along a different edge that we arrive on, so edges get used up in pairs.
`Backward direction`:
Suppose all degrees are even and consider a maximal trail. Because the degrees are even, it must be a circuit. If it used all edges, we are done. Otherwise, there exists $e = uv \in E(G)$ not used and having an end vertex u that is always visited. Cycle this vertex to be the first vertex of the circuit, and then extend the trail by using e. $\lightning$ □

5.6 Some Graph Invariants

Definition 5.8: Girth

The **girth**, $\text{girth}(G)$ of G is the least k such that G has a C_k subgraph.

Comment 5.5

If G is a forest, we say $\text{girth}(G) = \infty$.

Definition 5.9: Clique number

The **clique number**, $\omega(G)$, of a graph G is the greatest k such that G has a K_k subgraph.

Comment 5.6

K is necessarily induced. A graph K_ℓ is called a **clique**.

Definition 5.10: Independence Number

The **independence number**, $\alpha(G)$, of a graph G is the greatest k such that G has an induced $\overline{K_k}$ subgraph.

Comment 5.7

This is saying the complement of G, $\overline{G}$, has a clique in it.

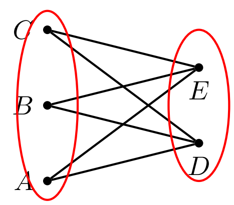
Result 5.1

Independence number is essentially the max number of vertices that are pairwise edgeless.

Definition 5.11: Complete Bipartite Graph

The **complete bipartite graph** $K_{m,n}$ (not the Kneser) is a graph with two sets of vertices, U and V, where all edges between two sets are all present, and no edges within each set itself.

Example 5.9: $K_{3,2}$



Definition 5.12:

We define $\Delta(G)$ as

$$\Delta(G) = \text{max vertex degree} = \max\{k : G \text{ has } K_{1,k} \text{ subgraph}\}$$

we also define $\delta(G)$ as

$$\delta(G) = \text{min vertex degree}$$

Definition 5.13: Proper Colouring

A **proper colouring** of G is a function $c : V(G) \to [k]$ such that if $uv \in E(G)$, then $c(u) \neq c(v)$.

Comment 5.8

k is the number of colours.

Definition 5.14: Chromatic Number

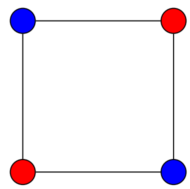
The **chromatic number** $\chi(G)$ of G is the min k such that a proper colouring exists.

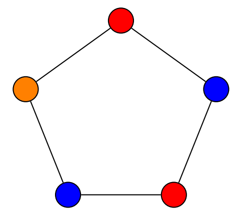
Example 5.10

Maps with neighbouring countries have different colours; putting people into graoups; or meetings into timeslots.

Example 5.11

We have $\chi(C_4) = 2$ and $\chi(C_5) = 3$.





In general, we have

$$\chi(C_n) = \begin{cases} 2 & \text{if } n \equiv 0 \pmod 2 \\ 3 & \text{if } n \equiv 1 \pmod 2 \end{cases}$$

5.6.1 It's hard to check k-colouring

Theorem 5.7

For general graph G, checking if G has a proper k-colouring is NP-complete.

Discovery 5.3

We have

$$\chi(G \sqcup H) = \max\{\chi(G), \chi(H)\}$$

Discovery 5.4

We have

$$\chi(G) = 1 \iff V(G) \neq \varnothing \wedge E(G) = \varnothing \iff G = \overline{K_k} \text{ for } k > 0$$

Definition 5.15: Bipartite

If $\chi(G) = 2$, we say that G is **bipartite**.

Proposition 5.4

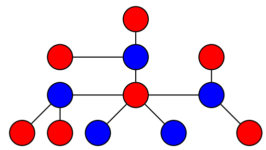
Every tree is bipartite.

Proof. Fix a vertex $x \in V(T)$ and colour it purple. Every $y \in V(T)$ has a unique path to x because it is a tree. Colour y purple if the path is even and green if the length is odd. Hence the colour set

$$\left|\{\texttt{purple}, \texttt{green}\}\right| = 2$$

is sufficient.

Example 5.12



□

Theorem 5.8

G is bipartite if and only if G contains no odd cycle.

Proof. `Forward direction:`
Odd cycles require three colours.
`Backward direction:`
STP for connected graphs. Suppose G is connected and has no odd cycles. Consider a spanning tree of G. It is bipartite so we are able to colour it with two colours.
Claim: this is a proper 2-colouring for G:
Suppose for a contradiction that there exist $x, y \in V(G)$ with $xy \in E(G)$ and $c(x) = c(y)$. Consider the unique x, y-path P in T. It has an even number of edges. Put together P and xy makes an odd cycle in G. $\lightning$ □

Discovery 5.5

We have

$$\chi(G) \leq |V(G)|$$

Proof. Everyone gets their own colour. □

Proposition 5.5

We have

$$\chi(G) \leq \Delta + 1$$

Proof. Colour the vertices in any order giving each vertex the smallest available colour. When I go to colour v, at worst it has $\Delta(G)$ neighbours and at worst they are all different colours, so I have at least 1 colour available.

Comment 5.9

This is the **greedy algorithm**.

□

Example 5.13

We have $\chi(\text{Petersen}) = 3$.

Example 5.14: Crown Graph

The crown graph, $♛_n$, has vertex set

$$V = \{u_1, \ldots, u_n, v_1, \ldots, v_n\}$$

where edges are $u_i v_j$ for $i \neq j$. By construction, we know that $♛_n$ is bipartite.

Comment 5.10

By the greedy algorithm, we might use as many as n colours, or as little as 2 colours.

5.6.2 Szekeres-Wilf 1968

Theorem 5.9: Szekeres-Wilf 1968

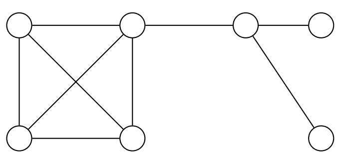
We have

$$\chi(G) \leq \max\{\delta(H) : H \subseteq G\} + 1$$

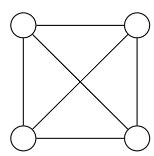
which is a little clever greedy algorithm.

Lecture 22 - Wednesday, October 30

Example 5.15



Graph G.



Graph $H \subseteq G$.

Notice that we have $\delta(G) = 1$ and $\delta(H) = 3$.

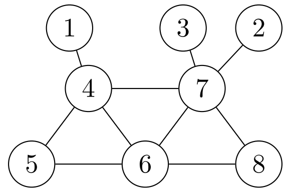
Example 5.16

For any non-trivial forest F, we have

$$\max\{\delta(H) : H \subseteq F\} = 1$$

by Proposition 4.2 (5.1). Then Szekeres-Wilf theorem (5.9) says that $\chi(F) \leq 2$.

Proof. Order the vertices $v_1, v_2, \ldots, v_n$ by sequentially removing a vertex of min degree. As an example, we have



Now we run the greedy algorithm in reverse order (starting with v_n). When we are colouring v_i for some $i \in [n]$, we just need to look at the induced subgraph $v_i, v_{i+1}, \ldots, v_n$. By construction, v_i is a minimum degree in the subgraph, so at worst its neighbours in the subgraph use $\max\{\delta(H) : H \subseteq G\}$ colours. Hence we have a colour available for v_i. □

Proposition 5.6

We have

$$\chi(G) \geq \omega(G)$$

Proof. A K_k needs k colours, so a number (strictly) less than k of colours will not colour the graph. □

Theorem 5.10: Tutte

For all k, there is a graph with G_k with $\omega(G_k) = 2$ such that $\chi(G_k) \geq k$.

Comment 5.11

$\omega(G_k) = 2$ means the graph G_k is triangle-free.

Proof. Mycielski
Given a graph G, its Mycielskian $M(G)$ is constructed as follows:
For each vertex $v \in V(G)$, make 2 vertices v_1, v_2, then

$$V(M(G)) = \underbrace{\{v_1 : v \in V(G)\}}_{V_1} \cup \underbrace{\{v_2 : v \in V(G)\}}_{V_2} \cup \{z\}$$

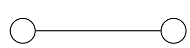
and

$$E(M(G)) = \{u_1v_1 : uv \in E(G)\} \cup \{u_1v_2 : uv \in E(G)\} \cup \{zv_2 : v \in V(G)\}$$

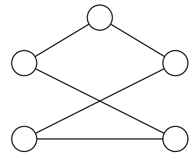
If G has n vertices and m edges, then $M(G)$ has $2n+1$ vertices and $3m+n$ edges.

Example 5.17: Grötzsch graph

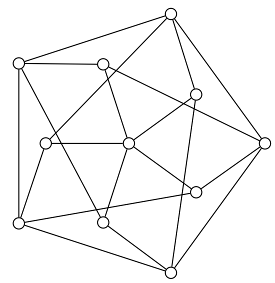
Here is an example of such construction, we have

$M_2 \quad =$ 

As a result of this, we have M_3 as follows:

$$M_3 = M(M_2) \qquad = $$ 

Now we have M_4 as

$$M_4 \qquad = $$ 

In fact, M_4 has a spacial name. It is known as the **Grötzsch graph**.

`Claim:` If G is not edgeless, then $\omega(G) = \omega(M(G))$, and also $\chi(M(G)) = \chi(G) + 1$

Lecture 23 - Friday, November 01

Since G has an edge, we have $\omega(G) \geq 2$. Note that a clique in $M(G)$ can contain at most one element of V'' sicne V'' is an independent set of G. Hence, a clique K_k with $k > 2$ cannot contain z. For any clique inside $V' \cup V''$, it looks like

$$v_1' v_2' v_3' \cdots v_k' v_\star''$$

Then we know $v_\star'$ is not in the clique since there is no edge $v_\star' v_\star''$ in $E(M(G))$.

Now we show $\chi(M(G)) > \chi(G)$. Suppose we have a proper k-colouring of $M(G)$. We may assume that z gets colour k. Then V'' just gets colours $1, \ldots, k-1$. For each $v' \in V'$, if it has colour k, change its colour to match V''. This is still a proper colouring of $V' = G$ (but maybe not of all $M(G)$), and it uses $k - 1$ colours.

No clique in $M(G)$ is strictly bigger than $\omega(G)$. Now given a proper colouring of G, copy the colours onto both V' and V'' and then make 1 new colour for vertex z. $\square$

5.7 What is $\chi(G)$, where G is theHadwiger-Nelson graph

Recall the Hadwiger-Nelson graph, say G, it has

$$V(G) = \mathbb{R}^2 \quad \text{and} \quad uv \in E(G) \text{ if } |u - v| = 1$$

If we tile the plane by hexagons with maximum diametre $= 0.99$, then we can 7-colour them. This is because there is no way for us to put a line of length 1 with a single hexagon. For a long time, people thought "maybe $\chi(G) = 4$". Then in 2018, Aubrey de Grey showed $\chi(G) \geq 5$ by finding a subgraph 1581 vertices with $\chi(G) = 5$.

6 Line Graph

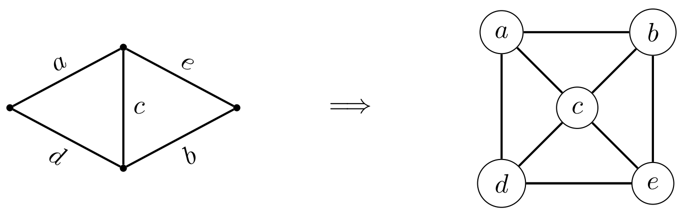
Definition 6.1: Line Graph

For $G = (V, E)$ being a graph, the **line graph** $L(G)$ of G has $V(L(G)) = E$ and $\{e, f\} \in E(L(G))$ if edges e, f are incident in G.

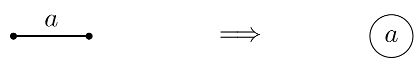
Example 6.1

Here are two examples:

1. Example 1:



2. Example 2:



Definition 6.2: Edge Chromatic Number

The **edge chromatic number** $\chi'(G)$ is $\chi(L(G))$.

Comment 6.1

We can think of this as colouring the edges of G, where incident edges get different colours.

Lecture 24 - Monday, November 04

Definition 6.3: Claw, $K_{1,3}$

The star $K_{1,3}$ is called the **claw**.

Discovery 6.1: Claw-free

If G contains no induced $K_{1,3}$, then we say G is claw-free.

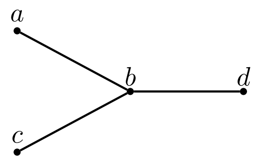
Claw-free graphs are very special.

6.1 Line Graphs are Claw-free

Proposition 6.1

For any G, $L(G)$ is claw free.

Proof. Suppose that $L(G)$ has an induced claw:

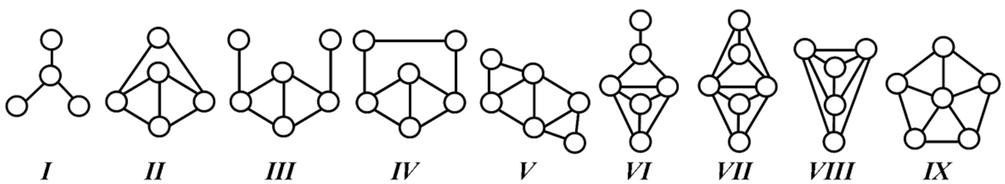


Then in G, edges a, c, and d are all incident to edge b, but they are not pairwise incident, which is impossible. □

6.1.1 Beineke 1970

Theorem 6.1: Beineke 1970

H is a line graph of some graph G if and only if H has none of the following as an induced graph:



Lemma 6.1

We have

$$\chi'(G) \geq \Delta(G)$$

Proof. $L(G)$ has a $K_{\Delta(G)}$ subgraph so $\omega(L(G)) \geq \Delta(G)$, but $\chi(L(G)) \geq \omega(L(G))$. □

Definition 6.4: Edge Clique Number

The **edge clique number** $\omega'(G)$ of G is $\omega(L(G))$.

Proposition 6.2

We have

$$\omega'(G) = \Delta(G)$$

unless $\Delta(G) = 2$ and G contains a C_3, in which case $\omega'(G) = 3$.

Proof. Exercise. □

Proposition 6.3

If $|V(G)| \geq 3$, then $M(G)$ has a claw, so it is not a line graph.

Proof. The vertex z is the center of the claw (see 5.6.2 for how we defined z). □

6.1.2 Vizing 1964

Theorem 6.2: Vizing 1964

For a finite graph G, $\chi'(G) \leq \Delta(G) + 1$.

Corollary 6.1

We always have $\chi'(G) \in \{\Delta(G), \Delta(G) + 1\}$.

Theorem 6.3

Computing $\chi'(G)$ is NP-complete. Although it is easy to approximate.

Theorem 6.4

For an arbitrary graph G, then

$$P\Big(\chi'(G) = \Delta(G)\Big) = 1 - \varepsilon$$

where $\varepsilon = 1 - 0.\dot{9}$.

Comment 6.2

Odd cycles are counter examples.

Lecture 25 - Wednesday, November 06

Definition 6.5: Matching Number

The **matching number** $\alpha'(G)$ of G is $\alpha(L(G))$.

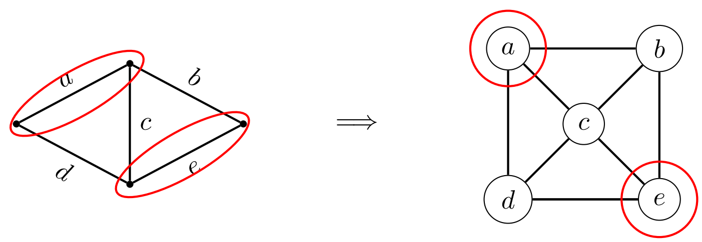
Definition 6.6: Matching

A **matching** in $G = (V, G)$ is the set of edges $M \subseteq E(G)$ such that $H = (V, M)$ has $\Delta(H) \leq 1$. Then $\alpha'(G)$ is the size of a largest matching.

Definition 6.7: Perfect

A matching M is **perfect** if every $v \in V$ has degree 1 in $H(V, M)$.

Example 6.2



Comment 6.3

Recall that computing $\alpha(G)$ is NP-complete, but computing $\alpha'(G)$ has a very fast algorithm.

Definition 6.8: Covering Number

The **covering number** $\beta(G)$ is the minimum number of vertices in G that cover all edges. Here, "covering" means "touch".

Proposition 6.4

For all graphs G, we have

$$\alpha'(G) \leq \beta(G)$$

As a result, if we have the equality holds, then the matching is maximal and the covering is minimal.

Proof. Let $X := \{(v, e) \in C \times M : v \in e\}$. We have

$$|X| = \sum_{(v,e)\in X} 1 = \sum_{v \in C} \deg_M(v) \leq \sum_{v \in C} 1 = |C|$$

and

$$|X| = \sum_{(v,e)\in X} 1 = \sum_{e \in M} |e \cap C| \geq \sum_{e \in M} 1 = |M|$$

because $\deg_M(v) \in \{0, 1\}$ and $|e \cap C| \in \{1, 2\}$. $\square$

Result 6.1

If $\alpha'(G) = \beta(G) = k$, we can determine both by finding a k-edge matching and a k-vertex cover.

Example 6.3: We don't always have $\alpha'(G) = \beta(G)$

An example here would be a pentagon, where its $\alpha' = 2$ and $\beta = 3$.

Discovery 6.2

If $|V(G)|$ is odd, G cannot have a perfect matching. Also note that $K_{1,k}$ does not have a perfect matching for $k > 1$.

6.1.3 Hall and Frobenius

Theorem 6.5: Hall

Let $G = (A \sqcup B, E)$ be a bipartite graph, then G has a matching covering all vertices of A if and only if for all $X \subseteq A$ we have $|N_G(X)|$, the number of neighbours of X in G, $\geq |X|$.

Corollary 6.2: Frobenius

G has a perfect matching if and only if for all $X \subseteq A$ we have $|N_G(X)| \geq |X|$ and $|A| = |B|$.

Lecture 26 - Friday, November 08

Proof. This is the proof for Hall Theorem.
`[Forward direction]`: This direction is clear.
`[Backward direction]`: Let $M \subseteq E$ be a maximal matching, let Z be the end points of M in A and Z' be the end points of M in B. If $A = Z$, then we have covered all A with M, so we are done. Otherwise, let $U := A \backslash Z \neq \varnothing$.

Definition 6.9: Alternating Path

An **alternating path** for M is a path in G that alternatively uses edges of M and $E \backslash M$.

Definition 6.10: Augmenting Path

An **augmenting path** is an alternating path beginning and ending at vertices not covered by M.

If we exchange matching/ non-matching edges along an augmenting path, we obtain a bigger matching, which contradicts the fact that M is our maximal matching. Thus M has no augmenting path. Let T' be the set of vertices in B reachable from U by an alternating path. Since there's no augmenting path $T' \subseteq Z'$. Let $T \subseteq Z$ be the elements paired with T'. Now, $N_G(T) \subseteq T'$ by the definition of T', and now $N_G(T) = T'$ by the definition of Neighbourhood. Also, by the definition of T', we have $N_G(U) \subseteq T'$. Hence $N_G(T \cup U) = T'$. However, we have

$$|T| = |T'| \qquad \text{and} \qquad U \neq \varnothing \qquad \text{and} \qquad U \cap T = \varnothing$$

because U are the vertices that aren't in the matching while T are those whom are in the matching. Thus

$$|T \cup U| > |T| = |T'| = |N_G(T \cup U)|$$

which is a contradiction. ↯ $\square$

6.1.4 König, Egervarg

Theorem 6.6: König, Egervarg

If G is bipartite, then $\alpha'(G) = \beta(G)$.

Proof. Consider the proof for Hall's Theorem. We showed that

$$N(T \cup U) = T'$$

Hence all edges are covered by $T' \cup (Z \backslash T)$, which has size $|T'| + |Z| - |T| = |Z| = \alpha'(G)$. Thus

$$\beta(G) \leq \alpha'(G)$$

which implies $\beta(G) = \alpha'(G)$. $\square$

6.2 Graph Products

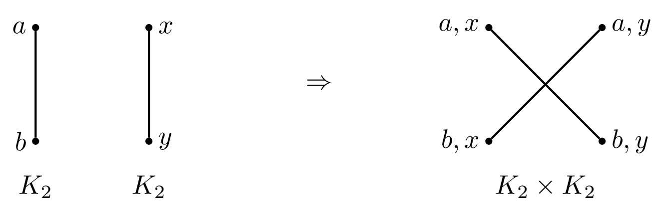
6.2.1 $G \times H$

Definition 6.11: $G \times H$

Let G and H be graphs, $G \times H$ has vertices $V(G) \times V(H)$ with edges

$$(g_1, h_1)(g_2, h_2) \in E(G \times H) \qquad \text{if} \qquad g_1 g_2 \in E(G) \text{ and } h_1 h2 \in E(H)$$

Example 6.4



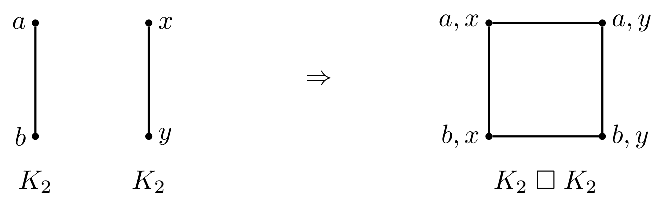
6.2.2 $G \,\square\, H$

Definition 6.12: $G \,\square\, H$

Let G and H be graphs, $G \square H$ has vertices $V(G) \times V(H)$ with edges

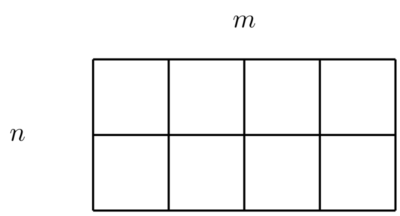
$$(g_1, h_1)(g_2, h_2) \in E(G \,\square\, H) \qquad \text{if} \qquad \begin{array}{ll} g_1 = g_2 \in V(G) \text{ and } h_1 h_2 \in E(H) & \text{or} \\ g_1 g_2 \in E(G) \text{ and } h_1 = h_2 \in V(H) & \end{array}$$

Example 6.5



Lecture 27 - Monday, November 11

Today we count perfect matchings, we look at $P_n \,\square\, P_m$:



Comment 6.4

How many perfect matchings are there in $P_2 \,\square\, P_n$?

Example 6.6

Proof. Fibonacci! $\square$

Comment 6.5

How about for $P_3 \,\square\, P_n$?

Discovery 6.3

We notice that there is no perfect matching if n is odd because in that case we would have an odd number of vertices.

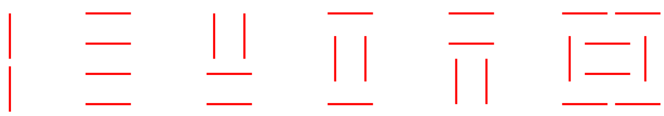
Proposition 6.5

$P_m \,\square\, P_n$ has a perfect matching if and only if at least 1 of m, n is even.

Proof. If both m and n are odd, then mn is odd, so no perfect matching. Otherwise, WLOG assume that m is even, it is easy to find a perfect matching. $\square$

Comment 6.6

We wonder how many perfect matchings there are for $P_4 \,\square\, P_n$.



etc.

Proof. Based on the pattern discovered above, one can work out a recurrence relation:

$$a_n = a_{n-1} + 5a_{n-2} + a_{n-3} - a_{n-4}$$

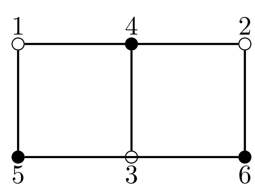
and then we can solve to get an exact formula with lots of $\sqrt{29}$. $\square$

Definition 6.13: Adjacency Matrix of Bipartite Graph

Let $G = (X \sqcup Y, E)$ be a bipartite graph with $|X| = |Y|$, then the adjacency matrix of G looks like:

$$A(G) = \begin{array}{cc} \begin{array}{cc} \text{X} & \text{Y} \end{array} & \\ \begin{bmatrix} \mathcal{O} & B \\ B^T & \mathcal{O} \end{bmatrix} & \begin{array}{c} \text{X} \\ \text{Y} \end{array} \end{array}$$

Example 6.7

$G \quad =$ 

For the above graph G, we have

$$A(G) = \begin{array}{cc} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} & \\ \begin{bmatrix} & & & 1 & 1 & 0 \\ & \mathcal{O} & & 1 & 0 & 1 \\ & & & 1 & 1 & 1 \\ 1 & 1 & 1 & & & \\ 1 & 0 & 1 & & \mathcal{O} & \\ 0 & 1 & 1 & & & \end{bmatrix} & \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} \end{array} \qquad \text{with} \qquad B = \begin{array}{cc} \begin{matrix} 4 & 5 & 6 \end{matrix} & \\ \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} & \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \end{array}$$

Recall for $C = (c_{ij})$ an $n \times n$ matrix, we have

$$\det(C) = \sum_{\omega \in S_n} \operatorname{sgn}(\omega) \prod_{i=1}^{n} c_{i,\omega(i)} \tag{Leibniz}$$

Definition 6.14: Permanent

The **permanent** of C is

$$\mathrm{per(C)} = \sum_{\omega \in \mathrm{S_n}} \prod_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{c_{i,\omega(i)}}$$

Example 6.8

For instance, for $C = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$, we have

$$\det(C) = 6 - 5 = 1 \qquad \text{and} \qquad \mathrm{per(C)} = 6 + 5 = 11$$

6.2.3 $\left|\texttt{perfect matching in } G\right| = \mathrm{per(B)}$

Proposition 6.6

For G bipartite and B as before, then

$$\left|\texttt{perfect matching in } G\right| = \mathrm{per(B)}$$

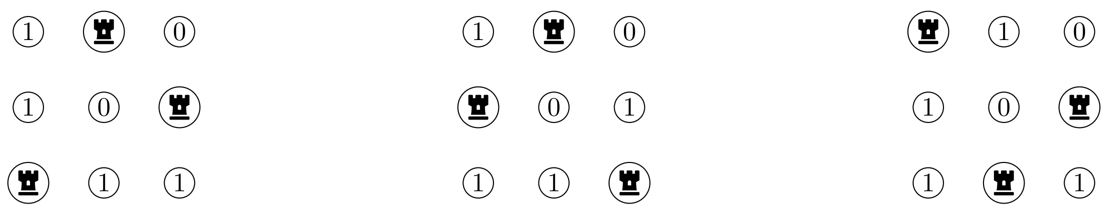
Proof. For each ω, we either get 0 or 1. We get 1 exactly if $(1, \omega(1))$, $(2, \omega(2))$, ..., all exist as edges, in which case they form a perfect matching.

Example 6.9

We have

$$\operatorname{per}\begin{pmatrix}1 & 1 & 0\\ 1 & 0 & 1\\ 1 & 1 & 1\end{pmatrix} = 3$$

Quick note: This is essentially counting the number of ways to place "rooks" on the 3×3 board replacing the 1's such that no two rooks are in the same row or the same column:



$\square$

Comment 6.7

Unfortunately, this proposition is pretty sad because permanent is hard to compute.

But our graphs aren't just bipartite, they are also planar.

7 Planar Graphs

7.0.1 Planar

Definition 7.1: Planar

G is **planar** if it can be drawn in the plane with no edges crossing each other.

Discovery 7.1

For G planar bipartite, we can obtain $\tilde{B}$ from B by adding certain signs so that

$$\mathrm{per(B)} = \mathrm{det(\tilde{B})}$$

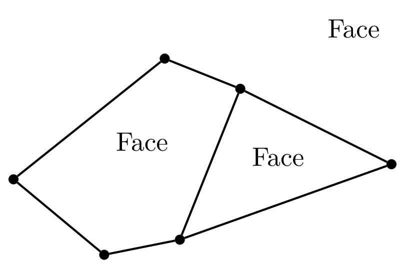
so we are happy, because determinants are eacy to compute.

Lecture 28 - Wednesday, November 13

7.1 Kasteleyn

7.1.1 Face

Definition 7.2: Face



Comment 7.1

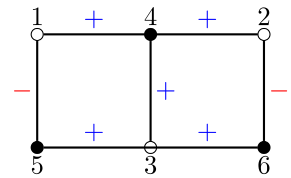
The outermost face is not bounded.

Theorem 7.1: Kasteleyn

Let G be a planar bipartite graph, then it is possible to assign signs to the edges of G so that every bounded face has an odd number of edges. In this case, the signed matrix $\tilde{B}$ satisfies

$$\#\text{ PM in } G = \mathrm{per(B)} = |\,\mathrm{det(\tilde{B})}|$$

Example 7.1

$G =$ 

Compute, we have

$$B = \begin{array}{c} \begin{array}{ccc} 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \quad \text{and} \quad \tilde{B} = \begin{array}{c} \begin{array}{ccc} 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \quad \Rightarrow \quad \det(\tilde{B}) = 3$$

Proof for grids $P_m \square P_n$:

We know there is a bijecton between the perfect matchings of G and the non-zero terms of the permanent. We just need to check that all the non-zero terms of the determinant have the same sign (so that there is no cancellation). As an example:

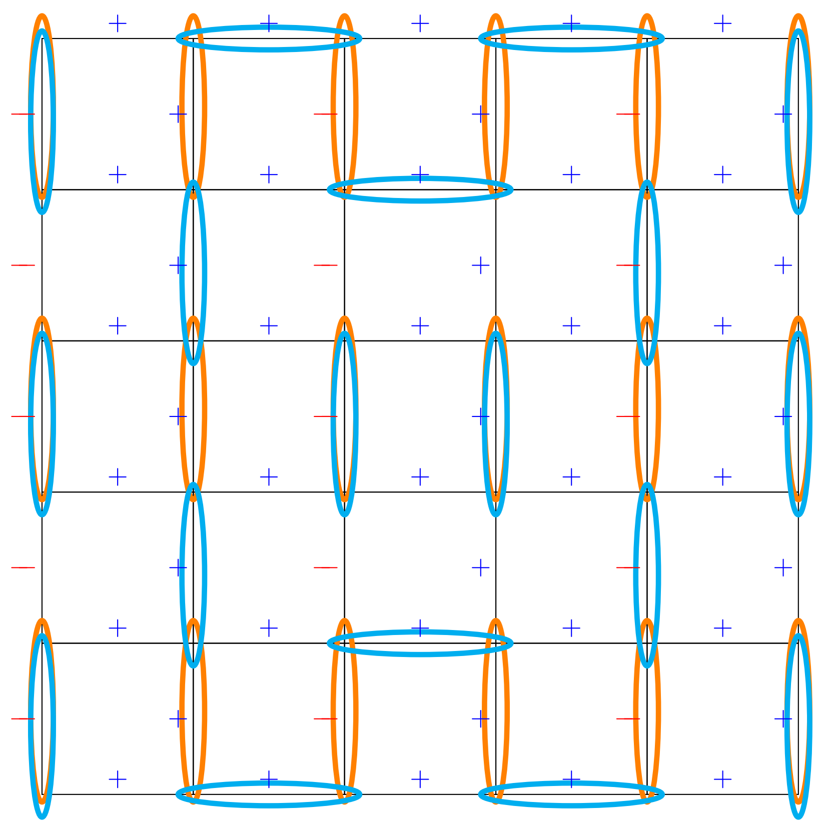


Figure 1: Example for $P_6 \square P_6$

Consider two perfect matchings and superimpose them. We get a union of doubled edges (double dimor con-

figurations) and disjoint cycles covering all the edges. Hence the perfect matchings are related by "rotating" each cycle. Therefore it is enough to compare two perfect matchings related by rotating a single cycle.
Let the edge be $v_1 v_2 v_3 \dots v_{2k}$, and define

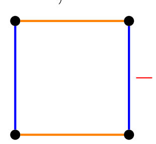
$$\pi = \frac{\prod_{i \text{ odd}} \mathrm{wt}(v_i, v_{i+1})}{\prod_{i \text{ even}} \mathrm{wt}(v_i, v_{i+1})}$$

[Claim: $\pi = (-1)^{k-1+\#\ \texttt{surrounded vertices}}$] where we define $\#$ surrounded vertices $:= \ell$.
We prove by induction on the area of cycle:

1. *Base Case:* We simply have a doubled edge, hence $\pi = 1$, $k = 1$, $\ell = 0$. We verify that

$$1 = \pi = (-1)^{1-1+0}$$

   In addition to that, consider the graph:



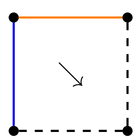
   We can easily see that $\pi = -1$, $k = 2$, $\ell = 0$, and thus

$$-1 = \pi = (-1)^{2-1+0}$$

   holds just fine.

2. *Induction:* Remove a corner box, WLOG assume the upper left one is removed. Now we have three cases:

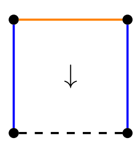
   (a) *Case 1:*



   Let π' be the sign for the smaller cycle, we have $\pi' = \pi$, $k' = k$, $\ell' = \ell - 1$. By induction we have

$$\pi' = (-1)^{k'-1+\ell'} \quad \Rightarrow \quad \pi = -\pi' = (-1)^{k-1+\ell}$$

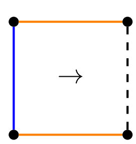
   (b) *Case 2:*



   We have $\pi = -\pi'$, $k = k' + 1$, $\ell = \ell'$. By induction we have

$$\pi' = (-1)^{k'-1+\ell'} \quad \Rightarrow \quad \pi = -\pi' = (-1)^{k-1+\ell}$$

   (c) *Case 3:*



   Basically the same calculation as above.

Lecture 29 - Friday, November 15

Example 7.2

Consider this example:

$$\tilde{B} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

When calculating the determinant, for this particular matching, it has signs:

- Sign for vertices: -1;
- Sign for position: -1

Comment 7.2

$\operatorname{sgn}(213) = (-1)^{\operatorname{inv}(213)} = (-1)^1 = -1.$

Last time we computed that the change in the sign from values was

$$\pi = (-1)^{k-1+\ell}$$

where the cycle has $2k$ edges and has ℓ vertices on the interior. This time we compute the signs arise from the positions. Now let our cycle be $u_1v_1u_2v_2\ldots u_kv_k$ with each edge $u_iv_i \in M$ and $u_iv_{i-1} \in M'$ (notice that u_1v_0 is the same as u_1v_k). We compute the two permutations.

$$\tilde{B} = \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & \cdots & v_k \end{array} \\ \begin{bmatrix} \bigcirc & & & & \square \\ \square & \bigcirc & & & \\ & \square & \bigcirc & & \\ & & \square & \bigcirc & \\ & & & \square & \bigcirc \end{bmatrix} \end{array} \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_k \end{array}$$

For M, it's the identity permutation, so it has sign $+1$. For M', it's $k123\cdots(k-1) = \omega$, so $\operatorname{sgn}(\omega) = (-1)^{\operatorname{inv}(\omega)} = (-1)^{k-1}$. But we claim that $\ell =$ even because everything is paired up. Thus

$$(-1)^{k-1} = (-1)^{k-1+\ell}$$

Thus, the signs match and cancel, so all the contributions to $\det(\tilde{B})$ have the same sign. $\square$

Now we can compute the number of perfect matchings in $P_m \,\square\, P_n$ as a determinant of a matrix with entries in $\{0, 1, -1\}$ (signed bipartite adjacency matrix). Enough to understand the eigenvalues of this matrix. Here, they come from eigenvalues of $A(P_k)$, which you can compute.

Corollary 7.1: Kasteleyn 1961

Assume $n =$ even, then the number of perfect matchings of $P_m \,\square\, P_n$ is

$$\prod_{k=1}^{n/2} \prod_{\ell=1}^{\lfloor m/2 \rfloor} \left(4\cos^2 \frac{\pi k}{n+1} + 4\cos^2 \frac{\pi \ell}{m+1} \right)$$

Example 7.3

For an actual 8×8 chessboard, there are 12988816 perfect matchings.

Kasteleyn has a similar result for any planar graph (not necessarily bipartite). We use Pfaffians instead of determinants.

Definition 7.3: Skew-symmetric

A matrix A is skew-symmetric if $A^T = -A$.

Comment 7.3

Notice that all diagonal entries of A have to be 0.

Discovery 7.2

A skew-symmetrix matrix $A \in \mathit{Matrix}_{odd \times odd}$ has determinant 0.

Proof. We have

$$\det(A) = \det(A^T) = \det(-A) = (-1)^{\#\ rows} \det(A)$$

$\square$

Otherwise, $\det(A)$ is always a perfect square:

$$\mathrm{Pf}(A)^2 = \det(A)$$

Example 7.4

$$\det\left(\begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}\right) = a^2$$

Example 7.5

$$\det\left(\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}\right) = 0 - abc + abc - 0 - 0 - 0 = 0$$

Lecture 30 - Monday, November 18

7.2 Euler's Theorem

Theorem 7.2: Euler's Theorem

If G is a connected planar graph with n vertices, m edges, and f faces, then

$$n - m + f = 2 = \chi(S^2)$$

Proof. If there is a cycle, delete an edge of it. Now $n' = n$, $m' = m - 1$ and $f' = f - 1$. One can reduce them to the case that G is a tree. We know that a tree has $m = n - 1$ and $f = 1$. Hence we have

$$n - m + f = n - (n - 1) + 1 = 2$$

as desired. $\square$

Discovery 7.3

Some complete graphs,

However, there is no way to draw K_5 planarly.

Corollary 7.2

If G is (WLOG) planar, then

$$m \leq 3n - 6$$

Proof. Let f_i be the number of faces with i sides. Then

$$f = \sum_{i=3}^{k} f_i$$

for some $k \in \mathbb{N}$, $k \geq 3$. However, we know that

$$2m = \sum i f_i \geq 3 \left(\sum f_i \right) = 3f$$

By Euler's Theorem, we have

$$\begin{aligned} n - m + f &= 2 \\ \Rightarrow \quad n - m + \frac{2}{3}m &\geq 2 \\ \Rightarrow \quad 3n - 6 &\geq m \end{aligned}$$

as desired. $\square$

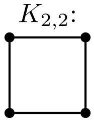
Discovery 7.4

K_5 is not planar because it has 10 edges, which is larger than $3 \cdot 5 - 6 = 9$.

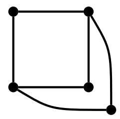
Example 7.6

Some complete bipartite graphs (as planarly as possible):

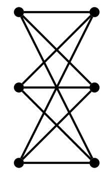
$K_{2,2}$:



$K_{2,3}$:



$K_{3,3}$:



However, there is no way for us to draw $K_{3,3}$ planarly, this is because it does not have triangles, thus in the proof of the above corollary, assuming $\sum if_i \geq 3 \left(\sum f_i\right)$ is not good enough.

Corollary 7.3

If G is (WLOG) planar and triangle-free, then

$$m \leq 2n - 4$$

Proof. As above, we have

$$2m = \sum if_i \geq 4 \left(\sum f_i\right) = 4f$$

By Euler's Theorem, we have

$$\begin{aligned} n - m + f &= 2 \\ \Rightarrow \quad n - m + \frac{1}{2}m &\geq 2 \\ \Rightarrow \quad 2n - 4 &\geq m \end{aligned}$$

as desired. □

Result 7.1

$K_{3,3}$ is not planar since it has too many edges as it should.

7.2.1 Subdivision

Definition 7.4: Subdivision

A **subdivision** of a graph G is given by adding some number of vertices of degree 2 along edges of G.

Comment 7.4

Clearly, G is planar if and only if all of its subdivisions are.

7.3 Kuratowski

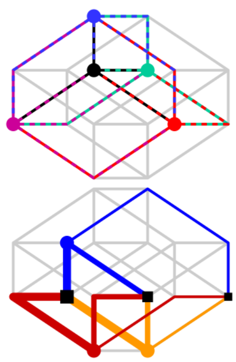
Theorem 7.3: Kuratowski

G is planar if and only if it does not contain any subdivision of K_5 or $K_{3,3}$.

Proof. **exercise**. □

Example 7.7

Found on Wikipedia:



Proof without words that a hypercube graph is non-planar using Kuratowski's or Wagner's theorems and finding either K_5 (top) or $K_{3,3}$ (bottom) subgraphs

7.3.1 Minor

Definition 7.5: Minor

A **minor** in a graph G is another graph M that can be obtained by

- deleting edges;
- deleting vertices;
- contracting edges.

7.4 Wagner

Theorem 7.4: Wagner

A graph G is planar if and only if it has neither a K_5 nor $K_{3,3}$ minor.

Lecture 31 - Wednesday, November 20

Discovery 7.5

If G has a subdivision of graph F, then G has a minor F.

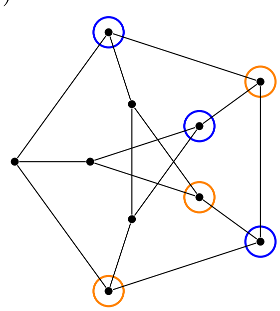
Proof. Take a subdivision. If we have a subdivision edge, we contract them. $\square$

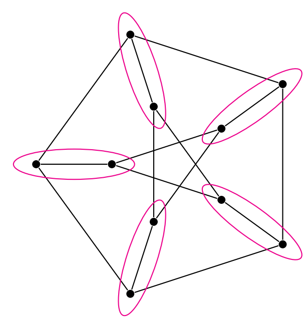
Result 7.2

Therefore Wagner's Theorem is a (not immediate) corollary of Kuratowski's Theorem.

Example 7.8

The Petersen is not planar because it has a $K_{3,3}$ subdivision (notice that it does't have K_5 subdivision because it does not have a single vertex of degree 4). However, it does have a K_5 minor (indeed a $K_{3,3}$ minor as well).





Comment 7.5

Not only isn't Petersen planar, but if one draw it in 3-space, one always get a pair of linked cycles.

7.4.1 Draw on any Surface Σ

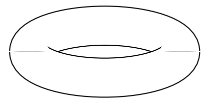
Theorem 7.5: Robertson-Seymour

For any fixed surface Σ, there is a finite list L of graphs characterizing the class of graphs drawable on Σ.

$$G \textit{ is drawable} \iff G \textit{ has no minor from } L$$

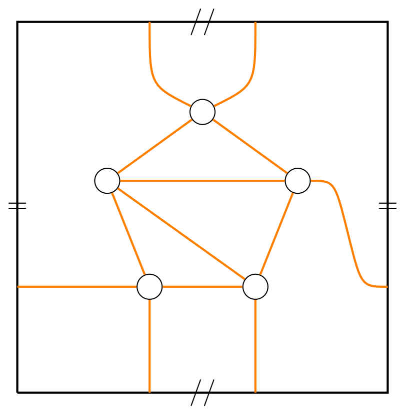
Comment 7.6

For the torus:

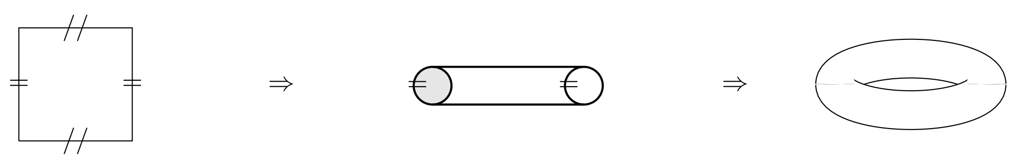


the list L has something like 18000(ish) entries and we do not know them all.

Example 7.9: Draw K_5 on torus so it's "planar"



where the pairs of sides marked // and = are glued together. Here is an illustration why the above picture is on a torus:



7.5 Four/ Five/ Six-Colour-Theorem

Comment 7.7

Guthrie started the conjecture in 1852. De Morgan proposes faking it as an axiom. Kempe 1879-1980 gives a false proof. Tait 1880-1891 also gives a false proof. Later on, computer assisted proof. Appel-Haken, 1976, wrote a >400 pages of hard cases analysis to reduce to checking 1834 configurations, 1000 hours of computing time. A simpler computer proof was given in 1996 by Robertson-Sanders-Seymour-Thomas.

Theorem 7.6: Four-Colour Theorem

If G is planar, then $\chi(G) \leq 4$.

Proof. Unfortunately we do not have enough time for the proof in this course. $\square$

Theorem 7.7: Six-Colour Theorem

If G is planar, then $\chi(G) \leq 6$.

Proof. Let H be any planar graph with n vertices and m edges. Then

$$m \leq 3n - 6$$

We also know that by Handshaking Lemma

$$2m = \sum_{v \in V(H)} \deg(v) \leq 6n - 12$$

Hence H must have a vertex of degree ≤ 5. Since this argument works with any subgraph of G, each one has a vertex of degree ≤ 5, so Szekeres-Wilf Theorem (5.9) tells us that $\chi(G) \leq 5 + 1 = 6$. $\square$

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Theorem 7.8: Sad Four-Colour Theorem

If G is planar **and triangle-free**, then $\chi(G) \leq 4$.

Proof. If H is planar and triangle-free, then $m \leq 2n - 4$. Hence

$$\sum_{v \in V(H)} \deg(v) = 2m \leq 4n - 8$$

which implies that there is a vertex with degree at most 3. Since this holds for every subgraph G, so Szekeres-Wilf Theorem (5.9) tells us that $\chi(G) \leq 3 + 1 = 4$. $\square$

Theorem 7.9: Grötzsch 1959

If G is planar and triangle-free, then $\chi(G) \leq 3$.

Proof. Too hard, so we skip. $\square$

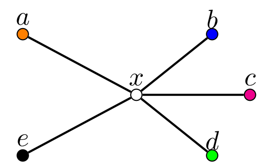
Theorem 7.10: Five-Colour Theorem, Heawood 1890, salvaging the false proof of Kempe

If G is planar, then $\chi(G) \leq 5$.

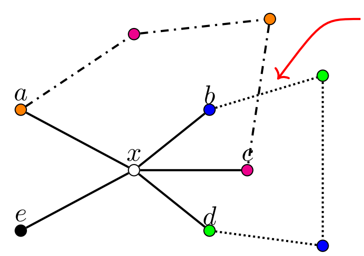
Proof. Proof by contradiction.
Let G be a minimal counterexample, so $\chi(G) > 5$ and G is planar, and every proper subgraph H of G has $\chi(H) \leq 5$ since G is the **minimal** counterexample. Recall that by Handshaking Lemma

$$2m = \sum_{v \in V(G)} \deg(v) \leq 6n - 12$$

Hence G must have a vertex of degree ≤ 5. In other words, $\delta(G) \leq 5$. Moreover, $\delta(G) \geq 5$ since $\chi(G) > 5$. In particular, if $\delta(G) \leq 4$, we can remove it, 5-colour the subgraph, and then add the vertex with minimal degree back. Thus $\delta(G) = 5$. Consider $x \in V(G)$ with $\deg(x) = 5$. We give a proper 5-colouring of $G \setminus \{x\}$. We know that the five neighbours of x must have 5 different colous or else we may just colour x with the missing colour.



Consider the vertices with the colours orange and pink. They induce a bipartite subgraph B. If a and c lie in different connected components, we may swap the colour of one of them to be the same of the other, and again we may colour x with the missing colour. Thus we know that B must be a path from a to c. Same argument gives a path between b and d, alternating their colours. But now observe that these two paths have to cross, contradicting that our picture is planar:



$\square$

8 Matroids

Lemma 8.1

If T is a spanning tree of G and e is NOT an edge of T, then $T + e$ has exactly 1 cycle C.
For any $e' \in C$, then $T + e - e'$ is also a spanning tree.

Comment 8.1

How is this related to Linear Algebra? (Think about basis).

Proof. Let $e = \{u, v\}$. A cycle in $T + e$ must use e because T itself is a tree which contains no cycle. Hence a cycle in $T + e$ consists of e and a u, v-path in T. But a tree has a unique path between any pair of vertices, thus there is only one cycle in $T + e$, say C. Let $e' \in C$. Deleting this edge from $T + e$ does not disconnect the graph since it is an edge in a cycle, so $T + e - e'$ is connected. It is also spanning since we didn't touch vertices at all. Notice that it has the same number of edges as T, so it is indeed also a tree. $\square$

Lemma 8.2

If T is a spanning tree of G and $e \in E(T)$, then $T - e$ has exactly two components.
If e' has an end point in each of the components, then $T - e + e'$ is a spanning tree.

Proof. Clearly $T - e$ has 2 components, say C_1 and C_2. Let $e' = \{u, v\}$ with $u \in C_1$ and $v \in C_2$. Then $T - e + e'$ is spanning and has a correct number of edges. For it to be a tree, it suffices to show that it is connected. Let $x, y \in V(T - e + e')$. If they are both in C_1 (or C_2), then there is a path in $T - e$. Otherwise, there is a path from x to u in C_1 and a path from v to y in C_2. Connect the two paths with edge e' yields us the path from x to y, as desired. $\square$

Lecture 33 - Monday, November 25

Comment 8.2

Notice that there exists an analogy between spanning tree in graphs and basis in vector spaces.

To make the analogy precise, we must invent matroids. Matroids were invented by

- Whitney, 1935 (topology)
- Nakasawa, 1935 (linear algebra)
- Mac Lane, 1936 (algebra/ geometry)
- Van der Waerden, 1937 (algebra)

8.1 Definitions of Matroids (lots of "cryptomorphic" definitions)

Definition 8.1: Matroid

Let E be a (finite) set and let $\varnothing \neq M \subseteq 2^E$, then M is a **matroid** if

- $X \in M$ and $Y \subseteq X$ together imply $Y \in M$;
- For X, Y maximal elements of M and $e \in X \setminus Y$, then there is $f \in Y \setminus X$ such that $X - e + f$ is a maximal element of M.
  Remark: Here, "maximal" means that $X + a \notin M$ for any $a \notin X$.

Definition 8.2: Independent Set & Dependent Set

Elements of M are called **independent sets**, while the other subsets of E are **dependent set**.

Definition 8.3: Basis & Circuits

Maximal elements of M are **bases**. Minimal dependent sets are **circuits**.

Definition 8.4: Rank

There is also a **rank** function, $r : 2^E \to \mathbb{Z}$, defined by

$$r(X) = \max\{|Y| : Y \subseteq X \text{ and } Y \in M\}$$

8.1.1 Vector Matroid

Example 8.1

E is a finite (multi)set of vectors in some vector space over some field $\mathbb{F}$. Let M be all subsets of E that are linearly indepent.
`Claim:` M `is a matroid.`

Proof. We check:

1. $M \neq \varnothing$ because $\varnothing \in M$;
2. If $X \in M$ is independent and $Y \subseteq X$, then Y is independent and so $Y \in M$;
3. A maximal element of M is a set of independent vectors that spans the space $\operatorname{span}(E)$.

Moreover, if X, Y both are maximal elements of M, then $|X| = |Y| = \dim \operatorname{span}(E) := d$. Hence, for $e \in X \setminus Y$,

$$\dim \operatorname{span}(X - e) = d - 1$$

then there exists $f \in Y \setminus X$ with $f \notin \operatorname{span}(X - e)$, and then $X - e + f$ is a d-element set that spans a d-dimensional space $\operatorname{span}(E)$, so it is another basis and so it is another maximal element of M. □

Comment 8.3

The example is called a **vector matroid**. A matroid realizable as a vector matroid with vectors over $\mathbb{F}$ is a **representable matroid** over $\mathbb{F}$.

Discovery 8.1

Some matroids are realizable over all fields, while some are realizable over no fields.

8.1.2 Cycle Matroid

Example 8.2

Let $G = (V, E)$ be a finite (multi)graph (with loops allowed, not necessarily connected). Let M be all subsets of E that induce forests X (i.e., deleting all edges outside X leaves a forest).
`Claim:` M `is a matroid`

1. $M \neq \varnothing$ because $\varnothing \in M$;
2. If $X \in M$, so X induces a forest and if $Y \subseteq X$,then Y induces a forest and so $Y \subseteq M$;
3. A maximal element of M is a "spanning forest", or spanning trees in each component of G. Let X, Y be maximal and suppose $e \in X \setminus Y$, then $X - e$ breaks one of the spanning trees into two components. But Y must have some edge $e' \notin X$ joining these two components, so now $X - e + e'$ is another maximal element.

Comment 8.4

The above example is called a **cycle matroid** of G. If M is a matroid realizable as a cycle matroid, we say M is **graphic**.

We notice the following connection:

`independent set`	$\longleftrightarrow$	`set of edges with no cycles`	(1)
`dependent set`	$\longleftrightarrow$	`set of edges with at least one cycles`	(2)
`basis`	$\longleftrightarrow$	`spanning tree in each component`	(3)
`circuit`	$\longleftrightarrow$	`cycle`	(4)
`rank of some collection of edges`	$\longleftrightarrow$	`max cardinality of an acyclic subcollection`	(5)

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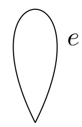
Definition 8.5: Loop

Let M be a matroid on E and let $e, f \in E$. Then e is a **loop** if $\{e\} \notin M$.

Example 8.3: Loop

In a cycle matroid:



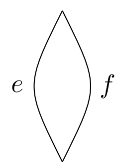
In a vector matroid: the zero vector.

Definition 8.6: Parallel

If neither e nor f is a loop, but $\{e, f\} \notin M$, we say e and f are **parallel**.

Example 8.4: Parallel

In a cycle matroid:



In a vector matroid: two vectors that are non-zero multiples of each other.

8.1.3 Free Matroid

Example 8.5: Free Matroid

Let E be a finite set and let $M = 2^E$. Then M is a matroid. The only basis is E and there are no dependent sets. M is called the **free matroid** on E. M is graphic, because it is the cycle matroid of a tree with $|E|$ edges. M is also representable over every field, take E to be any basis of $\mathbb{F}^{|E|}$.

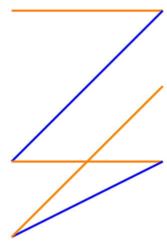
8.1.4 Transversal Matroid

Example 8.6: Transversal Matroid

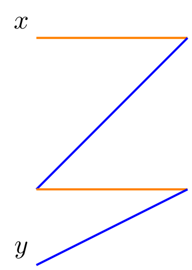
Let $G = (A \sqcup B, R)$ be a bipartite graph. Let M be those subsets I of A such that I can be covered by a matching. We claim that M is a matroid:

1. $\varnothing \in M$ so $M \neq \varnothing$;
2. If $J \subseteq I$ and there is a matching covering I, then there is a matching covering J, so $J \in M$;
3. Suppose I, I' are both maximal subset of A coverable by a matching. Consider $x \in I \setminus I'$ and take matchings covering I and I'.

If we have a path looks like



then this is an augmenting path for colour blue, which suggests that blue is not maximal. Hence we can only have



then swapping colours gives $I - x + y$, which is another maximal matching.

Comment 8.5

This is called **transversal matroid**.

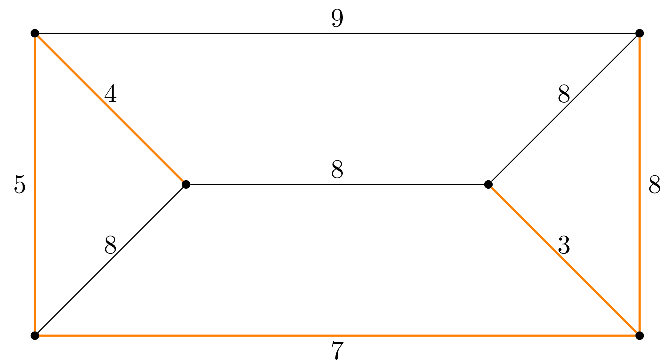
Discovery 8.2

The loop in a transversal matroid is a vertex in A with degree 0 (cannot match with anyone else). A pair of parallel edges in a transversal matroid are those with degrees exactly one and sharing the same neighbour.

8.2 Minimize the Weight on a Spanning Tree

Let G be a connected graph with a weight function $\omega : E(G) \to \mathbb{R}$. Our goal is to find a spanning tree of G that minimizes the total weight.

Example 8.7



The above spanning tree gives us a weight of 27.

8.2.1 Prim's Algorithm

Algorithm 8.1

Build a spanning tree T from nothing.

1. Choose any vertex $v \in V(G)$, set $T = v$.
2. While T isn't spanning,
   - look at all the edges joining a vertex of T to a vertex of $G \setminus T$;
   - let $e = xy$ be the cheapest one of these ($x \in V(T)$ and $y \in V(G \setminus T)$);
   - add y and e to the tree T.

Comment 8.6

This is a greedy algorithm.

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8.2.2 Kruskal's Algorithm (for min cost on spanning tree)

Algorithm 8.2

1. Order the edges by weight;
2. Make a spanning tree in each component of G by repeatedly adding the cheapest edge that does not create a cycle.

Theorem 8.1

Kruskal's Algorithm always makes a spanning tree in each component of min weight.

Proof. We may assume that G is connected. If we have built multiple subtrees in G, then there is an edge in G that connects two of them, so Kruskal will add such an edge. Thus we end with a spanning tree. Suppose Kruskal produces T and let $T*$ be a minimal weight spanning tree that shares the most edges with T. Our claim is that $T = T^*$. SFAC, let $e \in T \setminus T^*$ be the first such edge chosen by Kruskal, which indeed exists becuase both T and T^* are spanning trees, thus have the same number of edges. Then there exists $f \in T^* \setminus T$ with $T^* - e + f$ is a spanning tree. We must have $\omega(e) \geq \omega(f)$ by the minimality of T^*. However, by the fact that T^* shares as many edges with T as possible, we have $\omega(e) > \omega(f)$. Now it seems like Kruskal chose e even though f was cheaper, which is an oops. □

8.2.3 Can we apply Kruskal's Algorithm on Matroids?

Let M be a matroid on E and a weight function $\omega : E \to \mathbb{R}_+$. We want to find a basis of M with maximum weight.

Definition 8.7: Simplicial Complex

Let E be a finite set. A **simplicial complex** on E is a collection I of subsets of E such that

- $\varnothing \neq I$;
- If $X \in I$ and $Y \subseteq X$, them $Y \in I$.

So matroids are simplicial complexes satisfying an extra condition.

Given (E, I), I could have instead equivalently just specified B, its set of maximal elements, or F, all the subsets not in I, or C, or minimal elements of F. We could have called (E, I) instead (E, B), (E, C), or (E, F), or redundantly, (E, I, B, F, C).

Theorem 8.2

Let (E, I, B, F, C) be a simplicial complex, then TFAE:

(B) For $X, Y \in B$, if $e \in X \setminus Y$, then there exists $f \in Y \setminus X$ such that $X - e + f \in B$;

(B') For $X, Y \in B$, if $e \in X \setminus Y$, then there exists $f \in Y \setminus X$ such that $Y - f + e \in B$;

(U) For all $E' \subseteq E$, then if X, Y are maximal among subsets of E' that are in I, then $|X| = |Y|$;

(I) If $X, Y \in I$ with $|X| < |Y|$, then there exists $y \in Y \setminus X$ with $X + y \in I$;

(G) For all non-negative weight functions on E, the greedy algorithm finds a maximum weight independ set.

Proof. `(I)` $\Rightarrow$ `(U)`:
Fix $E' \subseteq E$ and let X, Y be maximal among subsets of E' that are in I. Now if $|X| < |Y|$, then there exists $y \in Y \setminus X \subseteq E'$ such that $X + y \in I$, so X wasn't maximal. ↯ Hence $|X| = |Y|$.

`(U)` $\Rightarrow$ `(I)`:
Let $X, Y \subseteq I$ satisfy $|X| < |Y|$. Set $E' = X \cup Y \subseteq E$, then X, Y are independent substes of E'. By (U), X is not maximal, so there exists some $y \in E'$ such that $X + y \in I$. But $E' \setminus X = Y \setminus X$, so (I) holds.

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`(B)` $\Rightarrow$ `(B')`:
By (B), all bases must be the same size. Otherwise, by repeatedly applying (B) to $X, Y \in B$ with $|X| < |Y|$, we can make $X \subsetneq Y$. This contradicts X a maximal independent set. Now fix $X, Y \in B$ and let $e \in X \setminus Y$. Apply (B) for each element of $X \setminus Y$ except e. Get a new basis X' that contains e, but all other elements are in Y. Let f be the unique element of $Y \setminus X'$, so $X' = Y + e - f$.
`(B')` $\Rightarrow$ `(B)`:
Again (B') implies all bases are the same size. Fix $X, Y \in B$ and $e \in X \setminus Y$. Apply (B') for each element of $X \setminus Y$ except e, we get a new basis Y' that contains all the elements of X except e. Let $f \in Y' \setminus X$, then $Y' = X - e + f$.
`(I)` $\Rightarrow$ `(B)`:
By (I), all bases are the same size. Let $X, Y \in B$ and $e \in X \setminus Y$. Now $X - e$ is independent and smaller than Y, hence (I) gives an element $f \in Y \setminus X$ that we can add to $X - e$ and remain independent. But by cardinality, $X - e + f$ must be a basis.
`(B')` $\Rightarrow$ `(G)`:
Some elements of E might have weight 0, so it might not be the case that every independent set of max weight is a basis. But at least some basis is an independent set of max weight. Run the greedy algorithm to get a basis X. Among bases of max weight, choose one Y that intersects X as much as possible. If $X = Y$, we are done. If $X \neq Y$, we will get a contradiction. Let $e \in X \setminus Y$ be the first greedily chosen element not in Y. By (B'), there exists $f \in Y \setminus X$ with $Y + e - f \in B$. If $\omega(e) < \omega(f)$, then Y isn't max weight. If $\omega(e) = \omega(f)$, then $Y + e - f$ also has max weight, contracting Y intersecting X the most. However, if $\omega(e) < \omega(f)$, our algorithm didn't chooce greedily.
`(G)` $\Rightarrow$ `(I)`:
Let $X, Y \in I$ with $h = |X| < |Y|$. We are looking for some $y \in Y \setminus X$ such that $x + y \in I$. We pick a useful weight function $\omega : E \to \mathbb{R}_+$. Define

$$\omega(e) = \begin{cases} h+2 & e \in X \\ h+1 & e \in Y \setminus X \\ 0 & e \notin Y \cup X \end{cases}$$

Since $X \in I$ and the elements of X are the highest weighted things, the greedy algorithm will first produce X. Now X is not a maximum weight independent set because its weight is $h(h+2) = h^2 + 2h$ whereas the weight of Y is at least $(h+1)(h+1) = h^2 + 2h + 1$. Hence the greedy algorithm can't be done. It must add another element of positive weight. But all such element are in $Y \setminus X$, so the greedy algorithm finds some $y \in Y \setminus X$ with $x + y \in I$. $\square$

Definition 8.8:

Let G be a graph. Then $\chi_G(q)$ is the function such that

$$\chi_G(k) = \text{ number of proper colouring of } G \text{ with at most } k \text{ colours}$$

Theorem 8.3

$\chi_G(q)$ is a polynomial with degree the size of a spanning tree $+1$, and it is divisible by q. Write

$$\frac{\chi_G(q)}{q} = w_0 q^r + w_1 q^{r-1} + w_2 q^{r-2} + \cdots$$

8.2.4 Conjecture: Read 1968

The sequence w_i is unimodal from any graph (matroid).

Theorem 8.4: Huh 2012

The above conjecture is in fact true for any matroid representable over $\mathbb{C}$, includes all graphs.

9 Practice

9.1 Enumeration

Exercise 9.1

Give bijective proofs of the following identity. For all $n \in \mathbb{N}$,

$$\sum_{k=0}^{n} \binom{n}{k} k = n2^{n-1}$$

Exercise 9.2

Give bijective proofs of the following identity. For all $n \in \mathbb{N}$,

$$\sum_{k=0}^{n} \binom{n}{k} k(k-1) = n(n-1)2^{n-2}$$

Exercise 9.3

For an integer $n \geq 1$, give a bijective proof that

$$\sum_{k \text{ even}} \binom{n}{k} = \sum_{k \text{ odd}} \binom{n}{k}$$

Proof. Add 1 if its not in the subset, remove it if it is. □

Exercise 9.4

Prove that

$$\sum_{m=k}^{n-1} \binom{m}{k} = \binom{n}{k+1} \qquad \text{for all } 0 \leq k < n$$

Proof. RHS is simply counting the number of ways of picking $k+1$ things to form a subset of $[n]$.
For the LHS, it describes another way of counting: we can first select the $(k+1)$-th largest element. Let's say this element is at position $m+1$, where m ranges from k (at least k elements before it to form a group of $k+1$) to $n-1$ (this is the maximum index it can take while still leaving room to have chosen k elements before it). Once we fix this $(k+1)$-th largest element at position $m+1$, we need to choose k elements from the first m elements. This can be done in $\binom{m}{k}$ ways. □

Exercise 9.5

Deduce that

$$3^n = \sum_{k=0}^{n} \binom{n}{k} 2^k$$

Proof. Let n be a positive integer. Let S_n be the set of all ordered pairs of sets (A, B) in which $A \subseteq B \subseteq \{1, 2, \ldots, n\}$. Let T_n be the set of all functions $f : \{1, 2, \ldots, n\} \to \{1, 2, 3\}$. Easy to see that $|T_n| = 3^n$. Moreover, we have bijection $g : S_n \to T_n$ defined as

$$g((A, B))(i) = \begin{cases} 3 & \text{if } i \in A \\ 2 & \text{if } i \in B, i \notin A \\ 1 & \text{if } i \notin B \end{cases}$$

where its inverse would be given by

$$A = \{i \in [n] : g(i) = 3\} \quad \text{and} \quad B = \{i \in [n] : g(i) = 3 \vee g(i) = 2\}$$

Therefore, the RHS is just counting the number of ordered pairs (A, B), while the LHS is counting the size of $|T_n|$. ☐

9.2 Generating Series

9.3 Linear Recurrence

9.4 Graphs

Exercise 9.6

Planar

Prove that every planar graph without a triangle (that is, a cycle of length three) has a vertex of degree three or less.

Proof. For triangle-free planar graphs, we have

$$\sum_{v \in V} \deg(v) = 2m \leq 4n - 8$$

by Corollary (7.3). Therefore, it has a vertex of degree three or less. ☐

Exercise 9.7

Bipartite

For n a positive integer, define the **prime graph** B_n to be the graph with vertex set $\{1, 2, \ldots, n\}$, where $\{u, v\}$ is an edge if and only if $u + v$ is a prime number. Prove that B_n is bipartite.

Exercise 9.8

Min Degree, Path, Cycle

Let G be a graph with minimum degree k, where $k \geq 2$. Prove that

1. G contains a path of length at least k;

2. G contains a cycle of length at least $k+1$.

Proof. `[part 1:]` Consider a maximal path $P = \{v_1, v_2, \ldots, v_\ell\}$ of G, where we know that v_1 has a degree of at least k. We know that all the k neighbours of v_1 has to be in P because otherwise P is not maximal. Therefore,

$$k \leq \ell - 1 = \text{length}(P)$$

`[part 2:]` We know that the cycle $C = \{v_1, v_2, \ldots, v_i, v_1\}$ has a length of $k+1$ for some $k \leq i \leq \ell$. □

Exercise 9.9

Complete Bipartite, Hamiltonian Cycle

Prove that the complete bipartite graph $K_{m,n}$ has a Hamilton cycle if and only if $m = n$ and $m > 1$.

Proof. Suppose that a complete bipartite graph $K_{m,n}$ is Hamiltonian. Then, it must have a Hamiltonian cycle which visits the two partite sets alternately. Therefore, there can be no such cycle unless the two partite sets have the same number of vertices. If $m = n = 1$, it is clear that $K_{m,n}$ contains no Hamiltonian cycle.

Conversely, it is obvious that the cycle $x_1, y_1, x_2, y_2, \ldots$ is a Hamiltonian cycle. □

Exercise 9.10

Regular, Girth

Prove that a k-regular graph of girth 4 has at least $2k$ vertices ($k \geq 2$).

Proof. Take two adjacent vertices in the graph, we know that they do not share a neighbour, so each of them have $n-1$ neighbours. In total, this gives us at least $2(n-1)+2 = 2n$ vertices. □

Exercise 9.11

Regular, Girth

Prove that a k-regular graph of girth 5 has at least k^2+1 vertices ($k \geq 2$).

Comment 9.1

The only values of k for which such a graph with exactly k^2+1 vertices can exist are $k = 2, 3, 7, 57$. This surprising result can be proved using elementary matrix theory (i.e., what you study in MATH 235). Examples are known for $k = 2, 3, 7$, but no example has yet been found for $k = 57$. Such a graph would have $57^2 + 1 = 3250$ vertices.

Proof. Take a vertex in the graph, it has k neightbours which are pairwise edgeless. Moreover, the neighbours' neighbours also cannot be adjacent because otherwise there are 4-cycle. Therefore, this gives us at least

$$1 + k + k(k-1) = k^2 + 1$$

vertices as desired. □

Exercise 9.12

Connected, Path

Prove that, if G is connected, any two longest paths have a vertex in common.

Proof. $\square$

Exercise 9.13

Connected, Components

If every vertex of a graph H with p vertices has degree at least $p/5$, prove that H cannot have more than 4 components.

Proof. Since every vertex has degree of at least $p/5$, we know that there are at least $p/5+1$ vertices in each component. Let c denote the number of components, so we have

$$p \geq c \cdot (p/5+1) \quad \Rightarrow \quad c \leq \frac{p}{p/5+1} = \frac{5p}{p+5} < 5$$

which implies that the number of components is upper bounded by 4. $\square$

Exercise 9.14

Bridge

Prove that a 4-regular graph has no bridge.

Proof. WLOG we may assume that the graph is connected because else we may simply consider a connected component of the graph G. SFAC that there is a bridge, so we may consider one of the component after deleting the bridge. (Can you finish the proof now?) $\square$

Exercise 9.15

Connected

Let G be a graph in which exactly two of the vertices u, v have odd degree. Prove that G contains a path from u to v.

Proof. If G is connected, then there exists a path from u to v. SFAC that G is not connected and u and v are not in the same components. Then we may reach a contradiction using the Handshake Lemma. We conclude that u and v has to be in the same component and thus there exists a path connecting them. $\square$

Exercise 9.16

Planar

Let G be a connected planar graph with p vertices and q edges and girth k. Show that

$$q \leq \frac{k(p-2)}{k-2}$$

Show also that if equality holds, all faces of G have degree k.

Proof. This is basically the same proof as corollaries (7.3) and (7.2). □

Exercise 9.17

Petersen, Planar, Kuratowski

Prove that the Petersen graph is non-planar without using any form of Kuratowski's theorem.

Proof. SFAC that Petersen is planar, and observe that girth for Petersen is 5. Therefore, we have

$$2m = \sum_i if_i = 5f \quad \Rightarrow \quad \frac{2}{5}m \geq f$$

which yields us

$$n - m + \frac{2}{5}m \geq 2 \quad \Rightarrow \quad 5n - 10 \geq 3m$$

For Petersen graph, it has 10 vertices and 15 edges, which is a contradiction to the above inequality. □

Exercise 9.18

Planar

Prove that if G is a planar graph in which every vertex has degree at least five, then $|V(G)| \geq 12$. Find such a graph with $|V(G)| = 12$.

Proof. By corollary (7.2), we know that

$$5n \leq \sum_{v \in V} \deg(v) = 2m \leq 6n - 12$$

so we must have $n \geq 12$. □

Exercise 9.19

Planar

Prove that every planar graph without a triangle (that is, a cycle of length three) has a vertex of degree three or less.

Proof. Check corollary (7.3). □

Exercise 9.20

Planar, Colouring

Prove that if G is a planar graph with girth at least six, then G is 3-colourable.

Proof. Easy to notice that the result is true when $|V| \leq 5$ since, in these cases, G has no cycles, and is thus bipartite and thus 2-colourable (and thus 3-colourable). So, we can assume $|V| \geq 6$. Then do induction. □

Exercise 9.21

Tree, Perfect Matching

Show that a tree has at most one perfect matching.

Proof. Suppose T has a perfect matching M. Let us build this perfect matching using the fact that if a vertex has degree 1, the edge incident to it is necessarily in M. We will build the matching while at the same time removing the edges that are not in the matching.

W know that every tree has at least two leaf vertices. Let x be a leaf and let y be its (only) neighbour. Then $xy \in M$. No edges incident to y can be in M, so let us remove them to get the graph T' (which may have multiple components). Each component of T' is a tree since it is a subgraph of T, so we repeat the process of adding an edge to M that is incident to one of the leaf vertices of T', and then deleting the surrounding edges. We repeat this process of building M until all vertices of T are saturated, and at each step, there is a unique choice for which edge to add for any given leaf vertex. Thus this construction yields a unique perfect matching for T. □

Exercise 9.22

Perfect Matching, Augmenting Path

Suppose that for some $n \geq 1$, graph G with p vertices satisfies $p = 2n$ and $\deg(v) \geq n$ for every vertex v. Prove that G has a perfect matching. (Hint: Prove that if M is a matching that is not perfect, then there exists an augmenting path of length 1 or 3.)

Proof. □

Exercise 9.23

Bipartite, Perfect Matching

(Difficult) Let G be a bipartite graph with bipartition A, B, where $|A| = |B| = n$, and suppose that every vertex of G has degree at least $\delta < n$. Prove that G has a matching of size at least the minimum of n and $(q - \delta^2)/(n - \delta)$.

Proof. Check https://math.stackexchange.com/questions/53252/difficult-bipartite-graph-problem. □

Exercise 9.24

Cover and Minimum Cover

Let G be bipartite with bipartition A, B. Suppose that C and C' are both covers of G. Prove that $\hat{C} = (A \cap C \cap C') \cup (B \cap (C \cup C'))$ is also a cover of G. In addition, prove that if C and C' are minimum covers, then so is $\hat{C}$.

Proof. □

Exercise 9.25

Matching

A deck of playing cards is arranged in a rectangular array of four rows and thirteen columns. Prove that there exist thirteen cards, no two in the same column and no two of the same value.

Proof. The problem is a bipartite matching problem, with one set being the ranks, and the other set being the columns. □

Exercise 9.26

Perfect Matching

Find a 3-regular graph having no perfect matching. (Such a graph must be nonbipartite.)

Proof. https://math.stackexchange.com/questions/3244938/a-trivalent-simple-graph-without-a-perfect-matching is one of the examples. □

Exercise 9.27

Permutation Matrix

An n by n permutation matrix is a matrix having one 1 and $n-1$ 0's in every row and in every column. Let N be an n by n matrix such that every row and every column contains k 1's and $n-k$ 0's. Prove that N is the sum of k permutation matrices.

Proof. □

9.5 Matroid

Exercise 9.28

Let m and n be two non-negative integers such that $m \leq n$. Let E be any set of n elements and define

$$\mathcal{I} := \{X \subseteq E : |X| \leq m\}$$

Show that $(E, \mathcal{I})$ is a matroid. This matroid is called the *uniform matroid* of rank m on an n-element set and is denoted by $U_{m,n}$.

Exercise 9.29

Show that the uniform matroid $U_{2,4}$ (see the above exercise for definition) is not graphic.

Exercise 9.30

Let $(E, \mathcal{I})$ be a matroid and $\mathcal{B}$ its set of bases. Define:

$$\mathcal{B}^* := \{E - B : B \in \mathcal{B}\}$$

Prove that $\mathcal{B}^*$ is the set of bases of a new matroid $(E, \mathcal{I}^*)$. This matroid is called the *dual matroid* of $(E, \mathcal{I})$.
Hint: Any collection of sets $\mathcal{B}$ is the set of bases of a matroid if and only if it satisfies the two following properties:

(B1) $\mathcal{B}$ is non-empty.

(B2) For any pair B_1, B_2 of members of $\mathcal{B}$ and any $x \in B_1 \setminus B_2$ there exists a $y \in B_2 \setminus B_1$ such that $(B_1 \setminus \{x\}) \cup \{y\} \in \mathcal{B}$.

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